THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY
T. A. A. BROADBENT, M.A.
ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E.10.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2

Vol. XXXVII, No. 319 FEBRUARY, 1953 5s. 6d. Net

CONTENTS

Intending members of the Mathematical Association are requested to communicate with one of the Secretaries, F. W. Kellaway, Miss W. Cooke, 29 Gordon Square, London, W.C.1. The subscription to the Association is 21s. per annum, and is due on Jan. 1st. Each member receives a copy of "The Mathematical Gazette" and a copy of each new Report as it is issued.

Change of Address should be notified to the Membership Secretary, Mr. M. A. Porter, Gordon House, 29 Gordon Square, London, W.C.1. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price. If change of address is the result of a change of appointment, the Membership Secretary will be glad to have notice of this.

Subscriptions should be paid to the Hon. Treasurer, Mathematical Association, Gordon House, 29 Gordon Square, London, W.C.1.

THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXXVII FEBRUARY, 1953 No. 319

PAIRS OF RELATED TRIANGLES STUDIED BY COMPLEX COORDINATES.

BY VICENTE INGLADA (Madrid).

Orthologic and metaparallel triangles.

Let $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ be the complex numbers corresponding to the angular points A_1, B_1, C_1 and A_2, B_2, C_2 of two related triangles, T_1, T_2, in a plane, and let homologous vertices be denoted by the same letters. The complex number

$$\delta_{\alpha\bar{\alpha}} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \bar{\alpha}_2 & \bar{\beta}_2 & \bar{\gamma}_2 \\ 1 & 1 & 1 \end{vmatrix} \qquad \text{.............................(1)}$$

is independent of the chosen systen of complex coordinates.* For, if the origin is changed, the formula for change of coordinates is $z' = z + \lambda$, and the new determinant has the value

$$\delta_{\alpha\bar{\alpha}}' = \begin{vmatrix} \alpha_1 + \lambda & \beta_1 + \lambda & \gamma_1 + \lambda \\ \bar{\alpha}_2 + \bar{\lambda} & \bar{\beta}_2 + \bar{\lambda} & \bar{\gamma}_2 + \bar{\lambda} \\ 1 & 1 & 1 \end{vmatrix} = \delta_{\alpha\bar{\alpha}},$$

while, for the rotation of the coordinate axes, the formula $z' = \tau z \; (\,|\,\tau\,| = 1)$ gives, in virtue of $\bar{\tau} = 1/\tau$,

$$\delta_{\alpha\bar{\alpha}}'' = \begin{vmatrix} \tau\alpha_1 & \tau\beta_1 & \tau\gamma_1 \\ \bar{\tau}\bar{\alpha}_2 & \bar{\tau}\bar{\beta}_2 & \bar{\tau}\bar{\gamma}_2 \\ 1 & 1 & 1 \end{vmatrix} = \delta_{\alpha\bar{\alpha}}.$$

The invariant $\delta_{\alpha\bar{\alpha}}$ depends on the areas S_1, S_2 of the triangles T_1, T_2 and on the areas of two triangles which are deduced from them. We call the first of the two latter, with area S_m and angular points the midpoints of the pairs of homologous vertices of T_1 and T_2, the *mid-triangle* of these. The other, the *second mid-triangle*, is the mid-triangle of T_1 and of the triangle derived from T_2 by rotation about any point of the plane through an angle of $\frac{1}{2}\pi$. Its area

* The number $\delta_{\alpha\bar{\alpha}}$ remains unaltered only when the orientation of the coordinate axes is preserved, one change of sense in this transforming $\delta_{\alpha\bar{\alpha}}$ to $\bar{\delta}_{\alpha\bar{\alpha}}$. The invariant depends, therefore, on the two triangles and the positive sense of rotation adopted in the plane.

is S_m'. Now the two identities

$$\begin{vmatrix} \alpha_1+\alpha_2 & \beta_1+\beta_2 & \gamma_1+\gamma_2 \\ \bar\alpha_1+\bar\alpha_2 & \bar\beta_1+\bar\beta_2 & \bar\gamma_1+\bar\gamma_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \bar\alpha_1 & \bar\beta_1 & \bar\gamma_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \bar\alpha_2 & \bar\beta_2 & \bar\gamma_2 \\ 1 & 1 & 1 \end{vmatrix} + \delta_{\alpha\alpha} - \bar\delta_{\alpha\alpha}$$

$$\begin{vmatrix} \alpha_1+i\alpha_2 & \beta_1+i\beta_2 & \gamma_1+i\gamma_2 \\ \bar\alpha_1-i\bar\alpha_2 & \bar\beta_1-i\bar\beta_2 & \bar\gamma_1-i\bar\gamma_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \bar\alpha_1 & \bar\beta_1 & \bar\gamma_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \bar\alpha_2 & \bar\beta_2 & \bar\gamma_2 \\ 1 & 1 & 1 \end{vmatrix} - i(\delta_{\alpha\alpha} + \bar\delta_{\alpha\alpha})$$

give, in fact,

$$\mathbf{R}\delta_{\alpha\alpha} = 8S_m' - 2(S_1+S_2), \qquad \mathbf{I}\delta_{\alpha\alpha} = 2(S_1+S_2) - 8S_m, \quad \ldots\ldots\ldots\ldots(2)$$

in virtue of the usual formula

$$S = -\frac{1}{4i}\begin{vmatrix} z_1 & z_2 & z_3 \\ \bar z_1 & \bar z_2 & \bar z_3 \\ 1 & 1 & 1 \end{vmatrix},$$

which expresses in complex coordinates the area of a triangle. The formulae (2) also provide a new proof of the invariance of $\delta_{\alpha\alpha}$ under the conditions given above.

Now let us determine the pairs of triangles which correspond to the cases in which $\delta_{\alpha\alpha}$ is real or purely imaginary. When it is real, we have *

$$\delta_{\alpha\alpha} = \alpha_1(\bar\beta_2 - \bar\gamma_2) + \beta_1(\bar\gamma_2 - \bar\alpha_2) + \gamma_1(\bar\alpha_2 - \bar\beta_2) = [r]. \quad \ldots\ldots\ldots\ldots(3)$$

The linear system

$$(z-\alpha_1)(\bar\beta_2 - \bar\gamma_2) = [r], \qquad (z-\beta_1)(\bar\gamma_2 - \bar\alpha_2) = [r], \qquad (z-\gamma_1)(\bar\alpha_2 - \bar\beta_2) = [r]$$

is therefore consistent, for (3) is just the result of elimination of z, by addition, from these three equations. But this system is equivalent to

$$(z-\alpha_1)/(\beta_2-\gamma_2) = [r], \qquad (z-\beta_1)/(\gamma_2-\alpha_2) = [r], \qquad (z-\gamma_1)/(\alpha_2-\beta_2) = [r]$$

and each equation represents a parallel from a vertex of T_1 to the respective side of T_2. The three lines therefore meet in a point and the triangles are *metaparallel*. Similarly, the condition

$$\delta_{\alpha\alpha} = [pi] \quad \ldots\ldots\ldots\ldots(4)$$

is equivalent to the consistency of the system

$$(z-\alpha_1)/(\beta_2-\gamma_2) = [pi], \qquad (z-\beta_1)/(\gamma_2-\alpha_2) = [pi], \qquad (z-\gamma_1)/(\alpha_2-\beta_2) = [pi].$$

Since these are the equations of the perpendiculars from the vertices of the first triangle to the corresponding sides of the second, the triangles are *orthologic* in this case. The formulae (2) then prove : *one-quarter the sum of the areas of the triangles is, for metaparallel triangles, equal to the area of the mid-triangle, and for orthologic triangles, equal to the area of the second mid-triangle.*

When $\delta_{\alpha\alpha} = 0$, both (3) and (4) are satisfied, and the triangles are at once orthologic and metaparallel.† But the vanishing of the determinant (1) requires that the elements of one row should be linear combinations of the elements in the other two rows, that is,

$$\alpha_1 = \lambda\bar\alpha_2 + \mu, \qquad \beta_1 = \lambda\bar\beta_2 + \mu, \qquad \gamma_1 = \bar\gamma_2 + \mu.$$

* We express the fact that a number z, which is in general complex, has in a particular instance a purely real value, without specifying that value, by writing $z = [r]$. Similarly, the fact that z is purely imaginary is denoted by writing $z = [pi]$.

† The pairs of homologous vertices are, besides, the same in the two correspondences in which the triangles are orthologic and metaparallel.

Since $z=\lambda\bar{z}'+\mu$ represents an inverse similarity, the triangles are now *inversely similar*. Noting also that (4) is satisfied by $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, $\gamma_1=\gamma_2$, we reach the theorem defining the orthocentre of a triangle.

The very simple form of conditions (3) and (4) enables us to give brief proofs of the properties of orthologic and metaparallel triangles. For example, from the identity

$$\delta_{\alpha\alpha}+\delta_{\alpha\beta}+\delta_{\alpha\gamma}\equiv\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \bar{\alpha}_2 & \bar{\beta}_2 & \bar{\gamma}_2\\ 1 & 1 & 1\end{vmatrix}+\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \bar{\beta}_2 & \bar{\gamma}_2 & \bar{\alpha}_2\\ 1 & 1 & 1\end{vmatrix}+\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \bar{\gamma}_2 & \bar{\alpha}_2 & \bar{\beta}_2\\ 1 & 1 & 1\end{vmatrix}=0,$$

we deduce that two triangles doubly orthologic (metaparallel) in cyclic fashion are triply orthologic (metaparallel). We call such triangles, respectively, *triorthologic* and *trimetaparallel*. Again, the identity (with l, m, n real numbers)

$$\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ l\bar{\alpha}_2+m\bar{\alpha}_3+n\bar{\alpha}_4, & l\bar{\beta}_2+m\bar{\beta}_3+n\bar{\beta}_4, & l\bar{\gamma}_2+m\bar{\gamma}_3+n\bar{\gamma}_4\\ 1 & 1 & 1\end{vmatrix}$$

$$=l\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \bar{\alpha}_2 & \bar{\beta}_2 & \bar{\gamma}_2\\ 1 & 1 & 1\end{vmatrix}+m\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \bar{\alpha}_3 & \bar{\beta}_3 & \bar{\gamma}_3\\ 1 & 1 & 1\end{vmatrix}+n\begin{vmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \bar{\alpha}_4 & \bar{\beta}_4 & \bar{\gamma}_4\\ 1 & 1 & 1\end{vmatrix}$$

proves that *if triangles $A_2B_2C_2$, $A_3B_3C_3$, $A_4B_4C_4$ are orthologic (metaparallel) to $A_1B_1C_1$, then so also is every triangle whose vertices A, B, C have equal areal coordinates (l, m, n) with regard respectively to the triangles $A_2A_3A_4$, $B_2B_3B_4$, $C_2C_3C_4$, that is, such that the quadrangles $AA_2A_3A_4$, $BB_2B_3B_4$, $CC_2C_3C_4$ are affine to each other.*

Mid-orthologic and mid-metaparallel triangles.

Now, let us take three real numbers, l, m, n, whose sum does not vanish, and put

$$\delta_1=(l\alpha_1+m\beta_1+n\gamma_1)/(l+m+n),\quad \delta_2=(l\alpha_2+m\beta_2+n\gamma_2)/(l+m+n). \quad\text{......(5)}$$

These complex numbers correspond to the points D_1, D_2 whose areal coordinates with respect to the triangles are l, m, n. The expression

$$\Phi_{\alpha\alpha}=l\alpha_1(\bar{\delta}_2-\bar{\alpha}_2)+m\beta_1(\bar{\delta}_2-\bar{\beta}_2)+n\gamma_1(\bar{\delta}_2-\bar{\gamma}_2)$$

is also independent of the coordinate axes. It is not changed, in fact, by the transformations $z'=\tau z$, $z'=z+\lambda$, since (5) gives

$$l(\bar{\delta}_2-\bar{\alpha}_2)+m(\bar{\delta}_2-\bar{\beta}_2)+n(\bar{\delta}_2-\bar{\gamma}_2)=0 \quad\text{..........................(6)}$$

Moreover, in virtue of

$$(l\bar{\alpha}_1+m\bar{\beta}_1+n\bar{\gamma}_1)\delta_2=(l+m+n)\bar{\delta}_1\delta_2=(l\alpha_2+m\beta_2+n\gamma_2)\bar{\delta}_1,$$

the conjugate

$$\Phi_{\alpha\alpha}=l\bar{\alpha}_1(\delta_2-\alpha_2)+m\bar{\beta}_1(\delta_2-\beta_2)+n\bar{\gamma}_1(\delta_2-\gamma_2)$$

takes the form

$$\bar{\Phi}_{\alpha\alpha}=l\alpha_2(\bar{\delta}_1-\bar{\alpha}_1)+m\beta_2(\bar{\delta}_1-\bar{\beta}_1)+n\gamma_2(\bar{\delta}_1-\bar{\gamma}),$$

that is, an inversion of the order of the triangles changes the invariant $\Phi_{\alpha\alpha}$ into its conjugate $\bar{\Phi}_{\alpha\alpha}$.

To obtain the geometrical condition satisfied by the triangles when $\Phi_{\alpha\alpha}$ is real, it is sufficient to note that the equations

$$(z-\alpha_1)/(\delta_2-\alpha_2)=[r],\quad (z-\beta_1)/(\delta_2-\beta_2)=[r],\quad (z-\gamma_1)/(\delta_2-\gamma_2)=[r]$$

represent the parallels through the vertices A_1, B_1, C_1 to the lines A_2D_2, B_2D_2,

C_2D_2; and multiplying these equations respectively by

$$l(\delta_2-\alpha_2)(\bar{\delta}_2-\bar{\alpha}),\quad m(\delta_2-\beta_2)(\bar{\delta}_2-\bar{\beta}_2),\quad n(\delta_2-\gamma_2)(\bar{\delta}_2-\bar{\gamma}_2),$$

adding and using (6), we have as the result of eliminating z,

$$\bar{\Phi}_{\alpha\alpha}=[r].$$

Thus a consequence of this condition is that the three parallels mentioned meet in a point. Likewise, if we remark that eliminating z from the equations

$$(z-\alpha_1)(\delta_2-\alpha_2)=[pi],\quad (z-\beta_1)(\delta_2-\beta_2)=[pi],\quad (z-\gamma_1)(\delta_2-\gamma_2)=[pi]$$

which define the perpendiculars from the vertices A_1, B_1, C_1 to the lines A_2D_2, B_2D_2, C_2D_2, we obtain

$$\Phi_{\alpha\alpha}=[pi],$$

we infer that the concurrence of these three perpendiculars corresponds to the case in which $\Phi_{\alpha\alpha}$ is purely imaginary. And, lastly, the equivalence of the pairs of conditions

$$\Phi_{\alpha\alpha}=[r],\quad \bar{\Phi}_{\alpha\alpha}=[r],\quad \text{and}\quad \Phi_{\alpha\alpha}=[pi],\quad \bar{\Phi}_{\alpha\alpha}=[pi],$$

proves the double theorem: *if two triangles are such that the parallels (perpendiculars) from the vertices A_1, B_1, C_1 of the first triangle to the lines A_2D_2, B_2D_2, C_2D_2 of the second are concurrent, then the parallels (perpendiculars) from the vertices A_2, B_2, C_2 of the second triangle to the lines A_1D_1, B_1D_1, C_1D_1 of the first are also concurrent.*

Let us denote the parallel and perpendicular relations between the triangles by the symbolic forms

$$A_1B_1C_1 \underset{lmn}{\parallel} A_2B_2C_2, \quad\text{and}\quad A_1B_1C_1 \underset{lmn}{\perp} A_2B_2C_2.$$

Then, if

$$\Phi_{\beta\alpha}=l\beta_1(\bar{\delta}_2-\bar{\alpha}_2)+m\gamma_1(\bar{\delta}_2-\bar{\beta}_2)+n\alpha_1(\bar{\delta}_2-\bar{\gamma}_2),$$
$$\Phi_{\gamma\alpha}=l\gamma_1(\bar{\delta}_2-\bar{\alpha}_2)+m\alpha_1(\bar{\delta}_2-\bar{\beta}_2)+n\beta_1(\bar{\delta}_2-\bar{\gamma}_2),$$

the identity

$$\Phi_{\alpha\alpha}+\Phi_{\beta\alpha}+\Phi_{\gamma\alpha}=(\alpha_1+\beta_1+\gamma_1)\{l(\bar{\delta}_2-\bar{\alpha}_2)+m(\bar{\delta}_2-\bar{\beta}_2)+n(\bar{\delta}_2-\bar{\gamma}_2)\}=0$$

shows that:

if $A_2B_2C_2 \underset{lmn}{\parallel} A_1B_1C_1$ *and* $A_2B_2C_2 \underset{lmn}{\parallel} B_1C_1A_1$, *then* $A_2B_2C_2 \underset{lmn}{\parallel} C_1A_1B_1$

if $A_2B_2C_2 \underset{lmn}{\perp} A_1B_1C_1$ *and* $A_2B_2C_2 \underset{lmn}{\perp} B_1C_1A_1$, *then* $A_2B_2C_2 \underset{lmn}{\perp} C_1A_1B_1$.

In the particular case $l=m=n=1$, the lines A_1D_1, B_1D_1, C_1D_1 become medians of the triangle T_1, the invariant $\Phi_{\alpha\alpha}$ takes the simpler form

$$\Phi'_{\alpha\alpha}=\tfrac{1}{3}\{(\alpha_1+\beta_1+\gamma_1)(\bar{\alpha}_2+\bar{\beta}_2+\bar{\gamma}_2)-3(\alpha_1\bar{\alpha}_2+\beta_1\bar{\beta}_2+\gamma_1\bar{\gamma}_2)\}$$

and the triangles are called *mid-metaparallel* $(\underset{111}{\parallel})$ and *mid-orthologic* $(\underset{111}{\perp})$ respectively.* The triangles which are mid-metaparallel or mid-orthologic in two cyclic ways are also triply so, and will be called, respectively, *trimid-metaparallel* and *trimid-orthologic*. But if we write

$$\alpha_1\bar{\alpha}_2+\beta_1\bar{\beta}_2+\gamma_1\bar{\gamma}_2=\mu_{\alpha\alpha},\quad \alpha_1\bar{\beta}_2+\beta_1\bar{\gamma}_2+\gamma_1\bar{\alpha}_2=\mu_{\alpha\beta},\quad \alpha_1\bar{\gamma}_2+\beta_1\bar{\alpha}_2+\gamma_1\bar{\beta}_2=\mu_{\alpha\gamma},$$

then it follows in virtue of

$$(\alpha_1+\beta_1+\gamma_1)(\bar{\alpha}_2+\bar{\beta}_2+\bar{\gamma}_2)=\mu_{\alpha\alpha}+\mu_{\alpha\beta}+\mu_{\alpha\gamma}$$

* *See* Agronomof, *Revista matemática hispano-americana*, 1927, pp. 299–304.

that
$$\Phi'_{\alpha\alpha}=\tfrac{1}{3}\{(\mu_{\alpha\beta}-\mu_{\alpha\alpha})+(\mu_{\alpha\gamma}-\mu_{\alpha\alpha})\}=\tfrac{1}{3}(\delta_{\alpha\beta}-\delta_{\alpha\gamma}),$$
$$\Phi'_{\alpha\beta}=\Phi'_{\gamma\alpha}=\tfrac{1}{3}\{(\mu_{\alpha\alpha}-\mu_{\alpha\beta})+(\mu_{\alpha\gamma}-\mu_{\alpha\beta})\}=\tfrac{1}{3}(\delta_{\alpha\gamma}-\delta_{\alpha\alpha}),$$
$$\delta_{\alpha\alpha}=\Phi'_{\alpha\gamma}-\Phi'_{\alpha\beta}, \qquad \delta_{\alpha\beta}=\Phi'_{\alpha\alpha}-\Phi'_{\alpha\gamma},$$
which prove that *the trimid-orthologic triangles compose the same class as the triorthologic triangles, and the class of trimid-metaparallel triangles is identical with that of the trimetaparallel triangles.*

Cosogonal triangles, and analogues.

The expression
$$\phi=(\alpha_1-\gamma_2)(\beta_1-\alpha_2)(\gamma_1-\beta_2)/(\beta_1-\gamma_2)(\gamma_1-\alpha_2)(\alpha_1-\beta_2) \quad \text{............(7)}$$
is, likewise, an invariant of the triangles T_1, T_2. For it is plain that it remains unaltered under the transformations $z'=z+\lambda$ and $z'=\tau z$, corresponding to translation of the origin and rotation of the coordinate axes respectively. This invariance is also a consequence of ϕ being the product of the three simple complex ratios $(\gamma_2\alpha_1\beta_1)$, $(\alpha_2\beta_1\gamma_1)$ and $(\beta_2\gamma_1\alpha_1)$. Further, since
$$(z\alpha_1\beta_1)(z\beta_1\gamma_1)(z\gamma_1\alpha_1)=1,$$
we can also obtain for this invariant ϕ the formula
$$\phi=(\gamma_2 z\alpha_1\beta_1)(\alpha_2 z\beta_1\gamma_1)(\beta_2 z\gamma_1\alpha_1), \quad \text{..................................(8)}$$
which expresses it as the product of three complex cross ratios.

By formula (8) and properties of the complex cross ratio we can now determine the nature of pairs of triangles corresponding to the particular values $\phi=[r]$, $\phi=[pi]$, and $|\phi|=1$ of the invariant (7). As the result of eliminating the variable z from
$$(\gamma_2 z\alpha_1\beta_1)=[r], \quad (\alpha_2 z\beta_1\gamma_1)=[r], \quad (\beta_2 z_1\gamma\alpha_1)=[r] \quad \text{........................(9)}$$
is just $\phi=[r]$, and since the equations (9) are complex equations of the circles $C_2A_1B_1$, $A_2B_1C_1$, $B_2C_1A_1$, these circles have, in the first case, a point in common. Similarly, the equations
$$(\gamma_2 z\alpha_1\beta_1)=[pi], \quad (\alpha_2 z\beta_1\gamma_1)=[pi], \quad (\beta_2 z\gamma_1\alpha_1)=[pi]$$
represent the circles through A_1, B_1; B_1, C_1; C_1, A_1, orthogonal respectively to the circles $C_2A_1B_1$, $A_2B_1C_1$, $B_2C_1A_1$, and when $\phi=[pi]$ these circles meet in a point. Finally, the equations
$$|(\gamma_2 z\alpha_1\beta_1)|=1, \quad |(\alpha_2 z\beta_1\gamma_1)|=1, \quad |(\beta_2 z\gamma_1\alpha_1)|=1$$
define the circles through the vertices C_2, A_2, B_2 with regard to which, respectively, the pairs A_1, B_1; B_1, C_1; C_1, A_1 are inverse. Thus if $|\phi|=1$, the three circles are coaxal.*

An inversion of the order of the triangles changes the invariant ϕ into
$$\phi'=(\alpha_2-\gamma_1)(\beta_2-\alpha_1)(\gamma_2-\beta_1)/(\beta_2-\gamma_1)(\gamma_2-\alpha_1)(\alpha_2-\beta)$$
thus each condition examined above,
$$\phi=[r], \quad \phi=[pi], \quad |\phi|=1$$

* $|\phi|=1$ implies the existence of one point common to the three circles, but as a consequence of this, the circles meet in a second point. For, if P is the first point common to the three circles, there exists another point Q through which the circles pass, since the equations
$$PA_1/PB_1=QA_1/QB_1, \quad PB_1/PC_1=QB_1/QC_1, \quad PC_1/PA_1=QC_1/QA_1$$
are certainly compatible. P and Q are, besides, the two points in the plane whose distances from the vertices A_1, B_1, C_1 are proportional to three given lengths.

has as a consequence an analogue in ϕ'. Hence, if the three circles of one of the above triads meet in a point, so do the three circles of the homologous triad obtained by permuting the triangles, and we have therefore proved :

When two triangles $A_1B_1C_1$, $A_2B_2C_2$ are such that the circles $C_2A_1B_1$, $A_2B_1C_1$, $B_2C_1A_1$ meet in a point, then so do the circles $C_1A_2B_2$, $A_1B_2C_2$, $B_1C_2A_2$

When two triangles are such that the three circles through the pairs of vertices A_1, B_1 ; B_1, C_1 ; C_1, A_1 orthogonal respectively to the circles $C_2A_1B_1$, $A_2B_1C_1$, $B_2C_1A_1$ meet in a point, so do the circles through A_2, B_2 ; B_2, C_2 ; C_2, A_2 orthogonal to $C_1A_2B_2$, $A_1B_2C_2$, $B_1C_2A_2$ respectively.

When the three circles through the vertices C_2, A_2, B_2, with regard to which the pairs, respectively, A_1, B_1 ; B_1, C_1 ; C_1, A_1 are inverse, belong to a pencil, then the circles through C_1, A_1, B_1, for which the pairs A_2, B_2 ; B_2, C_2 ; C_2, A_2 are respectively inverse, are also coaxal.

The triangles which satisfy the first condition have been called *cosogonal* triangles by the Roumanian mathematician Grunbaum. Since names for the other two cases are unknown to us, we shall say that the triangles satisfy conditions I and II respectively. And now, from the identity

$$\frac{(\alpha_1-\gamma_2)(\beta_1-\alpha_2)(\gamma_1-\beta_2)}{(\beta_1-\gamma_2)(\gamma_1-\alpha_2)(\alpha_1-\beta_2)}\,\frac{(\beta_1-\gamma_2)(\gamma_1-\alpha_2)(\alpha_1-\beta_2)}{(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)(\beta_1-\beta_2)}\,\frac{(\gamma_1-\gamma_2)(\alpha_1-\alpha_2)(\beta_1-\beta_2)}{(\alpha_1-\gamma_2)(\beta_1-\alpha_2)(\gamma_1-\beta_2)}=1$$

we deduce at once :

If two triangles are cosogonal, when related in two cyclic ways, they are also cosogonal when related in the third way (tricosogonal triangles).

If two triangles satisfy condition I when related in two cyclic ways, they are cosogonal when related in the third way.

If two triangles satisfy the condition II, when related in two cyclic ways, they do so when related in the third way.

V. I.

ANNUAL MEETING, SHEFFIELD, 1953

The Annual Meeting of the Mathematical Association for 1953 will be held at Sheffield, from Wednesday, 8th April to Saturday, 11th April. By the courtesy of the University of Sheffield, accommodation will be available for members at the University hostels, at an inclusive charge of £3 6s.

Wednesday, 8th April. 8 p.m., Reception by the University.

Thursday, 9th April. 9.15 a.m., Business meeting. 9.45 a.m., Presidential Address (Mr. K. S. Snell). 11.30 a.m., " Metallurgy " (Professor Quarrell). Afternoon visits, evening theatre.

Friday, 10th April. 9.15 a.m., " Infinity " (Professor Walker, Dr. Hamill). 11.30 a.m., " The Mathematics Division, N.P.L." (Dr. E. T. Goodwin). 2 p.m., " From Primary School to Secondary School ", (Mr. L. B. Birch, Mrs. E. M. Williams, Mr. M. W. Brown). 4.45 p.m., " Theory of Games " (Dr. S. Vajda).

Saturday, 11th April. 9.15 a.m., " From examination question to industrial problem " (Mr. J. F. Hinsley).

A detailed programme and a form of application for accommodation have been sent to all members, but further copies can be obtained from

Mr. C. R. Barwell, 53 Dukes Drive, Newbold, Chesterfield

to whom immediate application should be made by any member wishing to attend the meeting who has not yet registered.

MATHEMATICS: IN SEARCH OF A SOUL.*

BY J. E. BOWCOCK.

As this is to be a purely scientific talk it is best to start from fundamental definitions and concrete propositions.

So first the definitions.

Mathematics is merely the high-sounding name designated to the efforts of a child in its attempt to count up to ten, which is crystallised in those immortal words, " One little piggy went to market, one little piggy stayed at home. . . ." If this is not acceptable to some of the unenlightened *members*, the alternative " addin' up and takin' away " may be substituted. This is not however advisable, as some high authority might assert that such hypothetical insanities as " Einstein's Theory of Relativity " cannot be wholly explained by such simple operations. That is probably one of the failings of mathematicians that instead of simplifying problems down to basic realities they tend to overelaborate and either discard straightforward solutions as being stereotyped, or overlook them in the complexities of their unfathomable minds. I was always amazed at school when my maths master sometimes condemned a perfectly good straight-forward method because he had a pet method for solving that type of problem, which although it looked much prettier was much harder to understand and in many cases twice as long. Bertrand Russell also gives two definitions ; (1) Mathematics is that class of propositions in which P implies Q, where P and Q are propositions. (2) He also gives a much more feasible one, " Mathematics is the science where in which we do not know what we are talking about and do not care whether what we say about it is true." Mathematicians attempt to go farther asking " What is a number? " and in fact quite recently Peano put forward five axioms by which number might be defined. This is merely attempting to go right back to the start, but this is impossible because there is no beginning and no end of mathematics, so to cover their futility, they hide themselves under the name of " mathematical philosophers ", which is now taken to mean those who have contrived by devious means to keep out of their rightful place in the asylums.

Then we come to the definition of a soul. In view of present day research in the realms of atomic physics the classical view of the transparent figure in radiant white, floating upwards through the fourth dimensions, propelled by some unknown source of energy and propagating heavenly music through the ether by plucking the strings of a zither is no longer held to be valid. Its place has been taken by the presence of a body clad in colourful clothes and dazzling tie, rendering heavenly jazz—if such a thing exists—from a glittering platinum saxaphone. However this is a comparatively recent notion which has not yet been universally accepted and some of the old school still stick to the former idea as the only way in which theologians can explain eternity and the mathematician infinity. I'm not sure what the exact difference is but I'm sure a good dictionary will give it to at least three significant figures.

Well, I suppose most of you will be wondering what this is all about. To tell you the truth, I'm not quite sure, but the first concrete proposition I wish to put is that mathematics is the devil's own handywork and that it should no longer be called the Queen of Sciences but be renamed Mephistophelean mathematics. This will not be popularly acknowledged at this meeting as if it is followed to its logical conclusion, it means that you, some of the Highest Authorities on mathematics are, not, as often put forward, worthy and noble adventurers in a great cause but none other than the " Devil's Dis-

* A paper by a student member read at the Annual General Meeting of the North Staffordshire Branch.

ciples ". This will be quite a shock to many of you who have not thought of it in this light before, but if you will consider for a moment the number of really brilliant mathematicians who have turned to religion in their declining years you will realise that there is something in it. Mathematics is the craftiest contrivance Satan has ever yet devised. It is taught in schools, training colleges and universities all over the world and is looked on not as it should be, with scorn and derision, but with admiration. Granted that many school children look upon it with fear ; that is a natural, inborn faculty of a child for sensing and shrinking from anything that is evil, but mathematics is never looked upon as something which drags the soul of man deeper into the darkest depths of hell.

It is widely argued that teaching mathematics is valuable for the way in which it supports and develops the use of logic. This however is a fallacy. Even supposing that mathematics is logical, the logic used in it is so abstract that it is not in the least helpful to the ordinary thinker. Take for instance the examples by Lewis Carroll on logic, by which he states quite reasonable propositions and with the aid of logic proves nonsense. But Carroll is himself proved capable of being illogical as is shown in the *Mathematical Gazette* in which it quotes his treatment of the Cheshire Cat problem in which he says " . . . this time the cat vanished quite slowly, beginning with the end of its tail and ending with the grin which remained sometime after the rest of it had gone." It is obvious that by the time the tail had disappeared, the cat would be a Manx cat. This is a contradiction since the cat was by hypothesis a Cheshire one.

Also, by assuming a false proposition (by that I mean assuming a proposition and at the same time realising that it is not true) it is possible to prove anything. Suppose that we assume our chairman is Adolf Hitler, therefore the number of people who are our chairman and Adolf Hitler at the same time is one. But we all know that this is not so and that the number of people who are both at the same time is nil. Therefore $1=0$, now consider the number of pavements in London made of gold. This is obviously 0. But $0=1$. Therefore there is one pavement in London made of gold.

All through the ages the devil's hand has shown itself in some of the most noted mathematicians. The story of the Egyptian Ropestretchers was a notable example. Zeno for instance proved that motion is impossible and that Achilles would never catch the tortoise, failing to notice that this follows from the former. Imagine the effects of this on a party of pilgrims walking miles towards the Holy Land and then being told that they were getting nowhere. Then there was Epimenides of Crete who stated that all Cretans are liars, which ever since has filled the asylums with lunatics in search of the truth. Archimedes was a most immoral man, most noted for taking a bath and forgetting to dress afterwards. The devil was so pleased with Newton that instead of making him pluck the apple from the tree of knowledge, he made it fall on his head.

And so on all through the ages. The many Bernoullis, Euler, Lagrange, Laplace who invented equations which make electricity and hydro-dynamics much easier for those who don't have to solve them.

The power of mathematics instead of losing its grip as time passes is spreading. For instance, take a quotation, from Filson Young's book, " Shall I Listen? " in which he says, " There was a rather dreadful kind of music which darkened the eighteenth century and which might be called purely intellectual music, but to me music has never justified itself, what is interesting in it being much more completely achieved in the field of pure mathematics."

In view of our recent lecture on the " Twelfth root of two " (this title being merely a disguise to hide a piece of the devil's propanganda which attempted to disintegrate the beauty of Beethoven and Brahms into a mere piece of mathematics), it seems that this is a widely held view among mathematicians and that they have degenerated into such a state that they no longer hear music but only fundamental notes combined in some intricate function with the $\sqrt[12]{2}$ and overtones.

Another quotation from the same book referring to the B.B.C. time signal: " But the synchronous action of thought by an instantaneous signal makes time, in a curiously paradoxical way, a reality by abolishing it, and with it space—that other perhaps arbitrary and illusory dimension—is abolished also." Well, have you ever heard of a more devilish idea than that of abolishing time and space. It is almost on a parallel with the mathematicians' invention of what they call complex numbers which in reality is nothing more than " phoney " algebra. They have to invent Argand Diagrams, roots of negative numbers, transcendental equations, limits at infinity and other hallucinations all to lead the unsuspecting student off the straight and narrow path.

This talk was supposed to be on the students' point view, but I felt that as there were so many qualified teachers against the one or two comparatively innocent students (this 'innocent') supported by the fact that one physics lecturer said to us the other day, " I suppose that many of you are 'purer' mathematicians than I," that as the ratio M (teachers) to N (students) is very large, that to present the students' view openly might be taken to be a piece of pure conceit and that the minority has no right to say anything. So that what I have represented, if anything, is the exaggerated way in which we children see everything. With our being so mentally inferior, many things that appear small to your great minds, by the theory of relativity appear disproportionately large to us and that the playthings which you give us in the form of multiplication tables, simultaneous equations, the binomial theorem, vector analysis, differential equations, projective geometry, elliptic functions, etc., appear as the worst tortures of hell. But worst of all permutations and combinations.

We were fortunate to hear a revised version of Dr. Kuttner's talk on football pools which aroused so much interest at the British Association Meeting. It got so complicated that so far as many of us could see a team was playing and actually beating *itself*. Then Professor Hogben (Author of *Mathematics for the Million*) gave a talk at the Birmingham Branch Meeting on Permutations and Combinations in which he suggested that Pascal's triangle should be taught in the schools at nine or ten years of age, Perms and Combs following straight on at eleven and twelve. This seems an admirable idea, the one certain result being an increase in child mortality rates, cutting down the population and so solving the world food problem.

One thing ought to be said in conclusion, in order to justify the continued existence of mathematics. The one reason for its existence is to support several thousand men and women in the profit making concern of teaching. How else could people with such tortuous minds ever earn a living in this wicked world?

So please try to make the lessons as interesting as possible and be human beings instead of calculating machines and perhaps the inevitable, *i.e.* growing horns on your heads, will never happen.

J.E.B.

CATENARY GRAPHS AND THEIR USES.

By N. W. Honey

1. *The catenary.*

The curve formed by any flexible uniform chain supported at both ends, in equilibrium under the action of gravity, is the catenary

$$y = a \cosh x/a,$$

where a is the distance of the lowest point of the catenary above the horizontal x-axis. If the curve is required to pass through the origin, its equation becomes

$$y = a\,(\cosh x/a - 1)$$

and it is known that $a = H/w$, where H is the horizontal force applied at the ends of the catenary, and w is the weight per unit length of the chain. Thus

$$y = \frac{H}{w}\left(\cosh \frac{xw}{H} - 1\right). \quad \text{(i)}$$

The catenary formed by the chain is therefore completely defined when the ratio H/w is known, although the particular portion of the curve representing the chain is not.

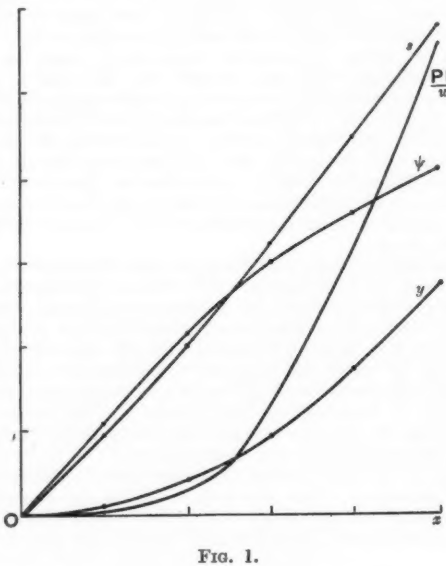


Fig. 1.

If we take $H/w = 1$, then equation (i) becomes

$$y = \cosh x - 1, \quad \text{(ii)}$$

and this curve is then plotted (Fig. 1). If now x and y in (ii) are multiplied by some constant K, then in order to satisfy (i) above, we must have $H/w = K$. Hence the curve in Fig. 1 represents all catenaries to some definite scale, and

the ratio of the full-size chain to its represented size on Fig. 1 gives the corresponding value of H/w. It should be noted that K has the dimension of a length. This curve, and others to be mentioned later, are plotted on squared paper, and results, except when x is very small, can with reasonable care be accurate within 1%. Some parts of the curves with small ordinate values can be plotted above the original curves with larger vertical scale to increase accuracy. The curves here shown in Figs. 1 and 2 are merely for purposes of illustration; the numerical examples use readings from the full-scale plotted curves.

Example. A very long chain has one end in the surface of the sea and the other end resting on the bottom 8 fathoms below and it is known that the chain is represented by that portion of the curve (ii) between $x=0$ and $x=0{\cdot}4$ (methods for determining this are given later). It is required to find the value of H/w.

From the graph, at $x=0{\cdot}4$ we have $y=0{\cdot}081$. Hence $0{\cdot}081$ represents 8 fathoms.

Thus the multiplication factor

$$K = 8 \text{ fms}/0{\cdot}081 = 98{\cdot}7 \text{ fms}.$$

Hence $H/w = 98{\cdot}7$ fms., and if $w = 2$ cwts. per fathom, $H = 9{\cdot}87$ tons.

2. *The curve of* ψ.

The angle ψ is the angle which the tangent to the catenary makes with the horizontal. From (ii),

$$\tan\psi = \sinh x,$$

and a curve for ψ is plotted. Obviously, if we expand the diagram by multiplying x and y by a constant K, the values of ψ at corresponding points of the curve and the chain will be equal.

Example. The example in § 1, but it is not now known that the chain is represented by that part of the curve between $x=0$ and $x=0{\cdot}4$, but it is known that at the surface $\psi = 23\frac{1}{3}^\circ$.

From the curve of ψ, it is found that when $\psi = 23\frac{1}{3}^\circ$, $x=0{\cdot}4$ and $y=0{\cdot}081$, and then we proceed as before. Also the tension in the chain at the surface is $H\sec\psi$, since H is the horizontal component of the tension. The vertical component is equal to the weight of chain off the bottom. Then, in the example, the tension at the surface is

$$H\sec\psi = 9{\cdot}87 \sec 23\tfrac{1}{3}^\circ \text{ tons} = 10{\cdot}73 \text{ tons}.$$

3. *The curve of* s.

If s denotes the length of the catenary from the origin up to a variable point, then it is well-known that

$$s = \sinh x,$$

and this curve is plotted on the same diagram. When x and y are multiplied by a constant K, to give details of the full-size chain, the values of s are multiplied by K.

Example. In the preceding example, it is further required to find the length of chain off the bottom.

At $\psi = 23\frac{1}{3}^\circ$, or $x=0{\cdot}4$, we find $s=0{\cdot}411$. Hence in the chain, the length off the bottom is

$$0{\cdot}411K = 0{\cdot}411 \times 98{\cdot}7 \text{ fms.} = 40{\cdot}6 \text{ fms.}$$

4. *The curve of* s/y.

This curve is also easily plotted, since

$$s = \sinh x = 2\sinh\tfrac{1}{2}x\cosh\tfrac{1}{2}x,$$
$$y = \cosh x - 1 = 2\sinh^2\tfrac{1}{2}x,$$

and so

$$s/y = \coth \tfrac{1}{2}x.$$

Obviously, when x and y are multiplied by K, the value of s/y remains the same for corresponding points of the chain and the curve.

5. *Curve of PE/w.*

It is sometimes necessary to know the value of the potential energy (PE) of the chain, calculated from the level of the axis of x, so the values of PE/w are also plotted. Now PE depends on the length of the chain, proportional to K, and also on the height of the centroid of the chain above the axis of x, also proportional to K. Hence, if we know the value of PE/w for the curve, the corresponding value for the chain is obtained by multiplying by K^2. The equation of the curve is readily found, for

$$ds = \cosh x\, dx$$

and

$$\begin{aligned} PE &= \int_0^x K^2 wy\, ds \\ &= \int_0^x K^2 wy \cosh x\, dx \\ &= \int_0^x K^2 w (\cosh x - 1) \cosh x\, dx, \end{aligned}$$

whence

$$PE/w = K^2 (\tfrac{1}{4} \sinh 2x - \sinh x + \tfrac{1}{2}x).$$

The PE/w curve is plotted for $K = 1$, and the values for the chain are found by using a factor K^2. It will be noted that all the four curves are easily plotted by using a book of mathematical tables.

Example. In the preceding example, it is required to find the potential energy of the chain.

From the curve, when $x = 0{\cdot}4$, $PE/w = 0{\cdot}0114$.

Thus, for the chain,

$$\begin{aligned} PE/w &= 0{\cdot}0114 K^2 \\ &= 0{\cdot}0114 \times 98{\cdot}7^2 \text{ fms.}^2 \end{aligned}$$

and

$$\begin{aligned} PE &= 0{\cdot}0114 \times 98{\cdot}7^2 \text{ fms.}^2 \times 2 \text{ cwts./fms.} \\ &= 222 \text{ cwt./fms.} = 66{\cdot}6 \text{ tons/ft.} \end{aligned}$$

6. *Chain of raised scope.*

A raised scope is said to occur when a chain leaves the bottom of the sea at an angle, instead of leaving it tangentially.

Suppose that we are given (i) the length of the chain, S fathoms, (ii) the depth of water, D fathoms, and (iii) the value of ψ at the surface, say ψ_1. Then from the graphs we can find x_1, y_1 and s_1. Also we know that the catenary up to the point (x_1, y_1) represents the chain to some at present unknown scale; but the multiplying factor $K = D/y_1$ fathoms, and hence the total length of the complete catenary to be tangential to the sea-bed must be Ds_1/y_1 fathoms, and $H/w = D/y_1$ fathoms.

The length on the bottom $= S - Ds_1/y_1$ fathoms, and if this is negative, that is, if $Ds_1 > Sy_1$, we have a raised scope, and the problem of finding the scale is resolved into that of finding the point (x_2, y_2) on the catenary such that $(y_1 - y_2)$ represents the depth of sea, D fathoms, to the same scale on which $(s_1 - s_2)$ represents the length of chain, S fathoms. If the scale factor is K fathoms, then

$$K(s_1 - s_2) = S \text{ fms.}, \quad K(y_1 - y_2) = D \text{ fms.}$$

Thus $$(s_1 - s_2)/(y_1 - y_2) = S/D, \quad \text{.............................(iii)}$$
and having found the point (x_2, y_2) we know K and hence H/w and ψ_2, the angle of escape. The point (x_2, y_2) may be found by trial and error on the diagram, but it may also be obtained by substituting the known values of $R\,(=S/D)$, s_1 and y_1, in the following formula :

$$y_2 = y_1 - \frac{Rs_1 - 1 - y_1}{\frac{1}{2}(R^2 - 1)}.$$

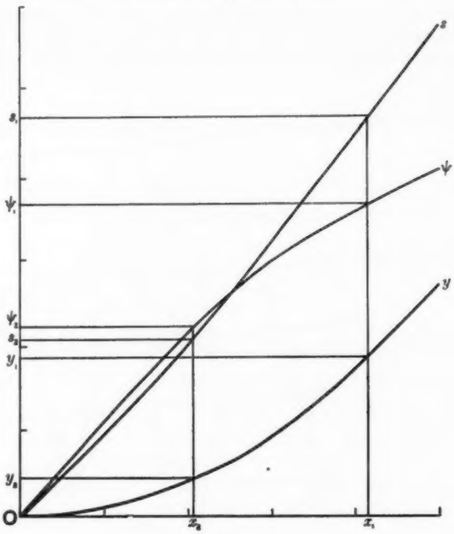


FIG. 2.

This formula can be obtained as follows. We have

$$\cosh x = 1 + y,$$
$$s^2 = \sinh^2 x = (1 + y)^2 - 1 = y^2 + 2y. \quad \text{.....................(iv)}$$

Now (iii) above can be written

$$s_1 - s_2 = R(y_1 - y_2),$$

which is clearly satisfied by $y_2 = y_1$. But from this equation

$$s_2^2 = (s_1 - Ry_1 + Ry_2)^2$$

or, from (iv), $$y_2^2 + 2y_2 = (s_1 - Ry_1)^2 + 2Ry_2(s_1 - Ry_1) + R^2 y_2^2,$$
that is,

$$(R^2 - 1)\,y_2^2 - 2y_2(1 - Rs_1 + R^2 y_1) + (s_1 - Ry_1)^2 = 0$$

a quadratic for y_2, of which we know that one root is y_1. From the sum of the roots, we have

$$y_1 + y_2 = 2(1 - Rs_1 + R^2 y_1)/(R^2 - 1),$$

which easily gives the above expression for y_2.

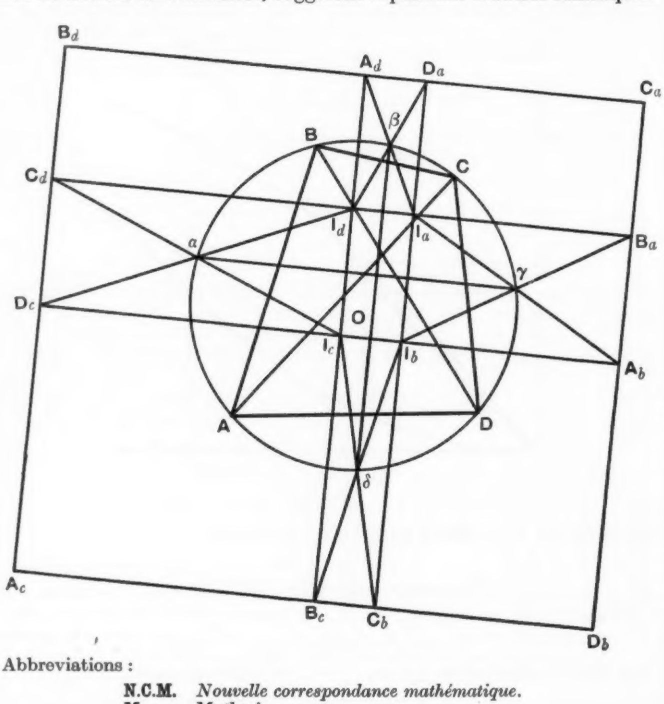
N. W. HONEY,
(Royal Corps of Naval Constructors)

SUR LE QUADRILATÈRE CONVEXE INSCRIPTIBLE.

PAR V. THÉBAULT.

UNE note de M. L. Droussent, (**M**, t. LVI, 93),* rappelle la propriété qu'ont les centres des cercles inscrits aux triangles ayant pour sommets ceux d'un quadrilatère convexe inscriptible, pris trois par trois, de coïncider avec les sommets d'un rectangle, dont l'origine paraît assez ancienne † et qui a été étendue aux seize centres des cercles tritangents des mêmes triangles.‡

Les différents groupes de points collinéaires ou concycliques signalés dans la note précitée, qui sont aussi mis en évidence dans les articles dont nous venons de donner les références, suggèrent cependant d'autres remarques.



* Abbreviations :

N.C.M. *Nouvelle correspondance mathématique.*
M. *Mathesis.*
I.M. *Intermédiaire des mathématiciens.*

† *Archives de Grunert*, 1842, 328, et **N.C.M.**, 1874, 199. Il' n'est pas impossible non plus que cette propriété ait été connue, dès l'année 1800, par l'auteur du théorème japonais dont il est question dans *Mathesis*, 1906, 257, et qui signale la relation $r_a + r_c = r_b + r_d$ entre les rayons des cercles (I_a), (I_b), (I_c), (I_d) dont la démonstration ne nous est pas parvenue.

‡ E. Lemoine, **N.C.M.**, 1878, 223, question 383, et J. Neuberg, **M.** 1906, 15 ; V. Thébault, *L'Education mathématique*, 28^{e} année, 1926, p. 128–9.

N.B.—C'est seize cercles tritangents qu'il faut lire et non vingt-quatre comme l'indique, par erreur, la note de M. Droussent.

1. **Notations.** Soient I_a, I_b, I_c, I_d les centres des cercles, de rayons r_a, r_b, r_c, r_d, inscrits aux triangles BCD, CDA, DAB, ABC dont les sommets forment les combinaisons trois à trois de ceux du quadrilatère convexe $ABCD$ inscrit à une circonférence (O), de rayon R ; (A_b, A_c, A_d), (B_c, B_d, B_a), (C_d, C_a, C_b), (D_a, D_b, D_c) les centres des cercles exinscrits dans les angles consécutifs de ces triangles pris dans l'ordre où on les a énumérés ; α, β, γ, δ et α', β', γ', δ' les milieux des arcs consécutifs AB, BC, CD, DA et leurs symétriques par rapport au point O ; ϵ et ϕ, ϵ' et ϕ' les milieux des arcs ABC, DAB et leurs symétriques par rapport au point O. Les diamètres $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$, $\epsilon\epsilon'$, $\phi\phi'$ sont les médiatrices des côtés AB, BC, CD, DA et des diagonales AC, BD du quadrilatère. Les points α et α', β et β', γ et γ', δ et δ', ϵ et ϵ', ϕ et ϕ' sont les centres de douze cercles qui passent deux à deux par les points A et B, B et C, C et D, D et A, A et C, B et D.

2. Observons, d'abord, que les segments rectilignes *

$$(C\phi I_a A_c,\ C\alpha' D_a D_b,\ C\delta' B_d B_a),\ (C\delta I_b B_c,\ C\alpha' D_a D_b,\ C\phi' A_d A_b),$$
$$(C\alpha I_d D_c,\ C\delta' B_d B_a,\ C\phi' A_d A_b),\ (C\phi I_a A_c,\ C\alpha I_d D_c,\ C\delta I_b B_c),$$

concourent au point C et portent les diagonales principales des hexagones †

$$(C_A) \equiv I_a D_a B_d A_c D_b B_a, \qquad (C_B) \equiv I_d D_a A_d B_c D_b A_b,$$
$$(C_D) \equiv I_d A_d B_d D_c A_b B_a,\ (C_{ABD}) \equiv I_a I_b D_c A_c B_c I_d,$$

dont *les côtés consécutifs sont rectangulaires.*

On obtient ainsi $4 \,.\, 4 = 16$ hexagones de cette forme *associés* quatre par quatre aux sommets A, B, C, D du quadrilatère fondamental.

3. **Théorème.** *Dans un quadrilatère inscriptible $ABCD$, les hexagones antipodaires d'un des sommets par rapport aux quatre hexagones à côtés consécutifs rectangulaires qui lui sont associés sont inscriptibles et leurs circonférences circonscrites se confondent avec la transformée de la circonférence circonscrite par l'homothétie de module 2 ayant pour centre le sommet considéré.*

L'hexagone antipodaire (C'_A) du sommet C du quadrilatère $ABCD$, par rapport à l'hexagone (C_A), a pour sommets successifs les points de rencontre C_1, C_2, C_3, C_4, C_5, C_6 des perpendiculaires en I_a et D_a, D_a et B_d, B_d et A_c, A_c et D_b, D_b et B_a, B_a et I_a aux droites CI_a et CD_a, CD_a et CB_d, CB_d et CA_c, CA_c et CD_b, CD_b et CB_a, CB_a et CI_a qui joignent le point C aux sommets successifs de l'hexagone (C_A). Ces points sont diamétralement opposés au point C sur les circonférences (β), (ϵ), (γ'), (β'), (ϵ'), (γ) ayant pour centres les extrémités des diamètres $\beta\beta'$, $\gamma\gamma'$, $\epsilon\epsilon'$ de la circonférence (O) perpendiculaires aux cordes CB, CD, CA aboutissant au sommet C.

Il en résulte déjà que l'hexagone (C'_A) est transformé de l'hexagone $(h) \equiv \beta\epsilon\gamma'\beta'\epsilon'\gamma$ par l'homothétie $(C, 2)$, de sorte qu'il est inscrit à une circonférence (C'), de rayon $2R$, centrée au point C' diamétralement opposé au point C sur la circonférence (O).

Mais, on constate aussi que les points C_1, C_2, C_3, C_4, C_5, C_6 coïncident avec les sommets des hexagones antipodaires (C'_B), (C'_D), (C'_{ABD}) du point C par rapport aux hexagones (C_B), (C_D), (C_{ABD}) et le théorème est démontré. Les sommets des quadruples d'hexagones antipodaires des points A, B, D par rapport aux hexagones à côtés consécutifs rectangulaires associés à ces points appartiennent aux transformées (A'), (B'), (D') de la circonférence (O) par les homothéties $(A, 2)$, $(B, 2)$, $(D, 2)$, égales à la circonférence (C').

* M, t. LVI, *loc. cit.*

† On affecte la lettre C des majuscules A, B, D, et ABD par analogie avec les minuscules a, b, d qui figurent en indices dans la désignation des droites CI_a, CI_b, CI_d qui portent une des diagonales principales des hexagones en cause.

Corollaire. *La circonférence* (O) *est tangente intérieurement aux circonférences* (A'), (B'), (C'), (D') *aux sommets* A, B, C, D *du quadrilatère* $ABCD$.

N.B. Le sommet C_1 commun aux hexagones (C'_A), (C'_B), (C'_D), (C'_{ABD}) est un septième point remarquable de la circonférence (β), … .

4. **Théorème.** *Dans un quadrilatère inscriptible* $ABCD$, *les aires des quatre hexagones à côtés consécutifs rectangulaires associés à un sommet sont inversement proportionnelles aux rayons des cercles tritangents du triangle formé par les sommets restant.*

L'aire C_A de l'hexagone (C_A) associé au sommet C du quadrilatère $ABCD$ équivaut à deux fois celle du triangle $C_aC_dC_b$ dont les sommets C_a, C_d, C_b coïncident, à la fois, avec les points de rencontre des côtés opposés de l'hexagone (C_A) et avec les centres des cercles exinscrits, de rayons r'_a, r'_d, r_b', du triangle ADB, en raison des égalités d'aires

$$\begin{aligned} C_aC_dC_b &= A_cD_bC_cB_d - (A_cC_bC_d + C_bD_bC_a + C_aB_bC_d) \\ &= \tfrac{1}{2}I_aD_aB_dA_cD_bB_a = \tfrac{1}{2}C_A. \end{aligned}$$

Or, le triangle ADB se confond avec le triangle orthique du triangle $C_aC_dC_b$ inscrit par suite à une circonférence de rayon $2R$, et, d'après une propriété connue, on a

$$\tfrac{1}{2}C_A = C_aC_dC_b = \tfrac{1}{2}(DB + BA + AD)\,.\,2R = (2R/r_c)\,.\,ADB \qquad \text{(1)}$$

Par analogie,

$$\begin{aligned} \tfrac{1}{2}C_B &= C_aI_cC_b = (2R/r'_d)\,.\,ADB, \\ \tfrac{1}{2}C_D &= C_aC_dI_c = (2R/r'_b)\,.\,ADB, \\ \tfrac{1}{2}C_{ABD} &= C_bI_cC_d = (2R/r'_a)\,.\,ADB, \end{aligned}$$

et le théorème est démontré.

Corollaire. *Les aires des quatre hexagones antipodaires correspondant aux hexagones à côtés consécutifs rectangulaires associés à un même sommet du quadrilatère inscriptible* $ABCD$ *sont inversement proportionnelles aux rayons des cercles tritangents du triangle formé par les sommets restant.*

En vertu du théorème 1, l'aire de l'hexagone antipodaire du sommet C du quadrilatère $ABCD$ équivaut à quatre fois l'aire h de l'hexagone (h). Or, les diagonales principales de celui-ci faisant entre elles des angles égaux ou supplémentaires à ceux du triangle ADB formé par les sommets restant, on a

$$h = R^2(\sin A + \sin D + \sin B) = \tfrac{1}{2}(DB + BA + AD)\,.\,R,$$

et, à cause des égalités (1),

$$h = \tfrac{1}{4}C_A\ ; \text{ d'où } C_A = 4h = C'_A = (4R/r_a)\,.\,ADB, \dots .$$

Corollaire. *L'aire d'un hexagone à côtés consécutifs rectangulaires associé à l'un des sommets du quadrilatère inscriptible* $ABCD$ *équivaut à celle de l'hexagone antipodaire correspondant.*

Note. Ces remarques sont des cas particuliers de théorèmes généraux relatifs a l'hexagone dont les côtés sont parallèles trois par trois, configuration que nous avons longuement étudiée.*

5. **Théorème.** *Dans un quadrilatère inscriptible* $ABCD$ *les forces représentées par les quarante-huit rayons des cercles tritangents des triangles* BCD, CDA, DAB, ABC *dirigés des centres de ces cercles vers leurs points de contact avec les côtés des triangles correspondants, ont une même résultante qui peut être représentée en grandeur et direction par quarante-huit fois le segment compris entre le centre du cercle circonscrit et le barycentre des sommets.*

* **M.**, 1932, supplément, 1937, 131 ; *Bulletin de l'Ecole Polytechnique de Bucarest* IX, 18–22 ; X, 33–38 ; XIII, 5–9.

Les forces en cause peuvent être reparties en quatre groupes comprenant celles que portent les rayons des cercles tritangents de chacun des triangles BCD, CDA, DAB, ABC.

Si les forces de chaque groupe admettent une résultante et si les quatre résultantes partielles ainsi obtenues ont à leur tour une même résultante (R), celle-ci se confondra avec la résultante de toutes les forces considérées.

Or, d'après un théorème de A. Boutin * récemment rappelé et généralisé par M. R. Goormaghtigh,† les quatre groupes de douze forces admettent chacun une résultante représentée en grandeur et direction par le quadruple des segments $\mathbf{OA'}$, $\mathbf{OB'}$, $\mathbf{OC'}$, $\mathbf{OD'}$ qui joignent le centre du cercle circonscrit (O) aux orthocentres des triangles BCD, CDA, DAB, ABC. Ces forces représentées par les segments $4\mathbf{OA'}$, $4\mathbf{OB'}$, $4\mathbf{OC'}$, $4\mathbf{OD'}$ étant concourantes, elles admettent une résultante qui se confond avec (R).

Mais les quadrilatères $ABCD$ et $A'B'C'D'$ sont symétriques et leur centre de symétrie O' coïncide avec le symétrique du centre O du cercle (O) par rapport au barycentre G des sommets du quadrilatère $ABCD$.‡ Le barycentre G' du quadrilatère $A'B'C'D'$ coïncide à son tour avec le symétrique du point G par rapport au point O', et la résultante (R) peut être représentée par le segment rectiligne

$$\mathbf{OX} = 4 \,.\, 4\mathbf{OG'} = 48\mathbf{OG},$$

porté par l'axe $\mathbf{OG}$ qui joint le centre du cercle circonscrit au barycentre des sommets du quadrilatère fondamental.

V. T.

GLEANINGS FAR AND NEAR

1723. At spring-tides, particularly when the line of the moon's apsides coincides with the syzygies, or when the ascending node is in the vernal equinox, or after heavy rains, the river still overflows its banks, and indicates its originally extended scite under ordinary circumstances.—Sir Richard Phillips, *A Morning's Walk from London to Kew* (1817), p. 198. [Per Mr. G. V. Groves.]

1724. I amused myself with a calculation of the probable number of persons who thus every day, between eight and six, pass to and from London within a distance of seven miles. In the present route I concluded the number to be something like the following, 200 from Pimlico, 300 from Chelsea, 200 from the King's Road and Sloane Street, 50 from Fulham and Putney, and 50 from Battersea and Wandsworth; making 800 per day. If then, there are twenty such avenues to the metropolis, it appears that the total of the regular ingress and egress will be 16,000 persons, of whom perhaps 8,000 walk, 2,000 arrive in public conveyances, and 6,000 ride on horseback, or in open or closed carriages. Such a phenomenon is presented no-where else in the world; and it never can exist except in a city which unites the same combined features of population, wealth, commerce, and the varied employments which belong to our own vast metropolis.—Sir Richard Phillips, *A Morning's Walk from London to Kew* (1817), p. 13. [Per Mr. G. V. Groves.]

1725. To braking there is a limit; mathematically it is g a deceleration of 32 feet per second, and modern brakes, plus a good road surface, already approach it as nearly as is safe for the occupants of a vehicle.—*Spectator*, Feb. 23, 1951, p. 238; article on "Mathematics of Road Safety". [Per Mr. N. M. Gibbins, Mr. A. W. Siddons.]

* **I.M.** 1915, 148. † **M.**, t. LV, 332. ‡ Mathot, **M.**, 1901, 25.

NOTE ON FUNCTIONAL RELATIONSHIP.

By G. Kreisel.

1. As far as I know, no definition has been given for the term " functional relation between functions $u(x, y)$ and $v(x, y)$ " which makes the following oft-quoted theorem true :

If

(1.1) u and v have continuous derivatives in a domain D, and on its frontier, then there is a functional relation between u and v in D, if and only if

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \text{ in } D. \tag{1.2}$$

An example shows that (1.2) is not sufficient to ensure that u is a one-valued, or finitely-valued, function of v, or *vice-versa* (that is, at points (x, y) where u takes some value u_0, v may take infinitely many values).

Divide the unit circle into annuli $r_n \leqslant r \leqslant r_{n+1}$, $r_n \to 1$, and define u and v in these annuli to satisfy (1.1) and (1.2), as follows : let a_n and b_n both converge to α, $a_n > a_{n+1} > b_{n+1} > b_n$, $(1 > \alpha > 0)$,

$$a_{n+1} - b_n < (r_{4n+4} - r_{4n+3})^3, \quad a_n - b_n < (r_{4n+2} - r_{4n+1})^3$$

and define
$$\begin{aligned} u(x, y) &= a_0, \quad \text{in} \quad 0 \leqslant r \leqslant r_1, \\ &= a_n, \quad \text{in} \quad r_{4n} \leqslant r \leqslant r_{4n+1}, \quad P_n, \\ &= b_n, \quad \text{in} \quad r_{4n+2} \leqslant r \leqslant r_{4n+3}, \quad P'_n, \end{aligned}$$

$$\frac{du}{dr} = -\frac{6(a_n - b_n)(r - r_{4n+1})(r_{4n+2} - r)}{(r_{4n+2} - r_{4n+1})^3} \quad \text{in} \quad r_{4n+1} \leqslant r \leqslant r_{4n+2}, \quad Q_n,$$

$$\frac{du}{dr} = \frac{6(a_{n+1} - b_n)(r - r_{4n+3})(r_{4n+4} - r)}{(r_{4n+4} - r_{4n+3})^3} \quad \text{in} \quad r_{4n+3} \leqslant r \leqslant r_{4n+4}, \quad Q'_n,$$

and continuous.

These five conditions are consistent, and u has continuous uniformly bounded differential coefficients in $r < 1$.

In P_n, P'_n, v is arbitrary, except that its differential coefficients are continuous, and zero on the boundary ; in Q_n and Q'_n, $v^n = u^{n+1}$.

Then (1.2) is satisfied identically in P_n and P'_n since there

$$\partial u/\partial x = \partial u/\partial y = 0,$$

and also in Q_n, Q'_n (by a simple calculation). But u takes the value α in each region Q_n, Q'_n, and $v = \alpha^{1+1/n}$ there ; v can take a whole range of values (in P_n, P'_n) when $u = a_n$ or $u = b_n$.

2. It is convenient to think of u and v as a mapping of D into a set R in the (u, v) plane, where the point (x, y) of D is mapped into $[u(x, y), v(x, y)]$ of R. A definition of " functional relation " will restrict the set R.

If one examines the usual sufficiency proof, which is based on the implicit function theorem, one finds that the argument applies in the neighbourhood, D_i say, of any point where the derivatives are not zero, say $\partial u/\partial x$ or $\partial u/\partial y$. In D_i, v is a one-valued differentiable function of u, that is, D_i is mapped into a differentiable curve in the (u, v) plane which is cut by a line $u = u_0$ in at most one point. But the argument fails at points where all derivatives vanish.* Note that since u and v are continuous in the closure of D, they are bounded, so that all sets in the (u, v) plane which we introduce are inside some (finite) square.

* The difficulties which we consider will not be encountered in simple examples, where one can picture the set of points where the derivatives vanish. Hence such difficulties may be properly ignored in school work.

The most natural proof of the necessity of (1.2) (for whatever form of theorem we may decide on) consists of showing that if (1.2) is false at (ξ, η) of D some neighbourhood of (ξ, η) is mapped into a square of the (u, v) plane, *i.e.*, into a region of non-zero area. Hence (1.2) is a necessary condition if our definition of functional relationship requires the set R to be of measure zero.

Now this is true of the set of curves into which the regions D_i are mapped since there is only an enumerable set of such curves. It turns out that the set (u, v) which is the image of the points of D where all derivatives vanish, has zero area (ordinary plane Jordan measure). Thus the theorem will be true if we use the

Definition : Functions $u(x, y)$, $v(x, y)$ are said to be functionally related for (x, y) in D, if the set $[u(x, y), v(x, y)]$ is bounded, consists of an enumerable infinity of differentiable curves of the form $v = V_i(u)$ or $u = U_j(v)$, and a set of Jordan measure zero.

We shall also show that $|dV_i/du| < (1+\epsilon)$, $|dU_j/dv| > (1+\epsilon)$, $\epsilon > 0$, if the functions are suitably chosen.

In the more detailed proofs that follow, (1.1) is assumed ; D is bounded.

3. We need one calculation, based on the implicit function theorem.

If $\partial u/\partial x \neq 0$, say $\partial u/\partial x \geqslant \alpha > 0$, at (x_0, y_0), and $u(x_0, y_0) = u_0$, then, under continuity of $\partial u/\partial x$, the implicit function theorem states that there is a differentiable function $x(u, y)$, defined near (u_0, y_0) such that $u[x(a, y), y] = a$ for a near u_0, and y near y_0 ; also

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y} = 0$$

and

$$\frac{\partial x}{\partial y} = -\frac{\partial u}{\partial y}\Big/\frac{\partial u}{\partial x}, \text{ possible since } \frac{\partial u}{\partial x} \neq 0.$$

Considering then the variation of $v(x, y)$ on a curve where u is constant,

$$\frac{\partial}{\partial y} v[x(u, y), y] = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\Big/\frac{\partial u}{\partial x}. \tag{3.1}$$

(3.2) Also
$$\frac{\partial}{\partial u} v[x(u, y), y] = \frac{\partial v}{\partial x}\Big/\frac{\partial u}{\partial x}.$$

Similar results hold if any one of the other derivatives is not zero at (x_0, y_0).

4. If (1.2) is false at (ξ, η), the standard theory of the transformation of coordinates in a double integral shows that a neighbourhood of (ξ, η) in D is mapped into a region of non-zero area. The result is also evident from (3.1) if one remembers that, by continuity,

$$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\Big/\frac{\partial u}{\partial x}$$

is different from zero in a whole region if it is not zero at (ξ, η).

5. Now, the set S_1, where

$$(1+\epsilon)\left|\frac{\partial u}{\partial x}\right| > \left|\frac{\partial v}{\partial x}\right| \quad \text{or} \quad (1+\epsilon)\left|\frac{\partial u}{\partial y}\right| > \left|\frac{\partial v}{\partial y}\right|$$

and the set S_2, where

$$(1+\epsilon)\left|\frac{\partial v}{\partial x}\right| > \left|\frac{\partial u}{\partial x}\right| \quad \text{or} \quad (1+\epsilon)\left|\frac{\partial v}{\partial y}\right| > \left|\frac{\partial u}{\partial y}\right|$$

are both open (and overlapping). Each point of these sets lies inside a neighbourhood where v is a one-valued differentiable function of u (in S_1, by (3.2),

$|\, dv/du \,| < 1 + \epsilon$) or u is a one-valued differentiable function of v (in S_2, with $|\, du/dv \,| < 1 + \epsilon$). By a familiar covering theorem (neighbourhoods consisting of circles with rational centres and rational radii), the sets S_1 and S_2 are covered by an enumerable infinity of such neighbourhoods, each of which is mapped into differentiable curves as required.

To deal with the set S of D where all derivatives vanish, choose η so small that the derivatives do not vary more than ϵ in any square of side η in D (where ϵ is given and positive). This is possible by uniform continuity of the derivatives in the closure of D. Now cover D by a (finite) net of squares of gauge η, parallel to the axes, and pick out those which contain a point of S. The total area of such squares is bounded by the area of D.

Since in each such square there is a point where the derivatives vanish (a point of S), the derivatives do not exceed ϵ in absolute value. Hence, if (x, y), (x', y') lie in one of these squares

$$|\, u(x, y) - u(x', y') \,| < 2\epsilon\eta, \quad |\, v(x, y) - v(x', y') \,| < 2\epsilon\eta,$$

and therefore the image set is covered by a square of side $2\epsilon\eta$ in the (u, v) plane. Thus the image set of S can be covered by squares of total area not exceeding $4\epsilon^2$ times the area of D. Since ϵ is arbitrary, and the number of squares used is finite, the set is of Jordan plane measure zero.

6. Paragraphs 4 and 5 prove the theorem with the definition of paragraph 2. Note that under the usual analogous conditions the same argument applies to n dimensions.

7. For readers familiar with Lebesgue measure, it is evident that there is another definition of functional relationship which, under condition (1.1), makes (1.2) equivalent to a functional relation between u and v. Let the set (u, v) be of Lebesgue measure zero. Yet not every set of Lebesgue measure zero consists of an enumerable set of differentiable curves, and a set of Jordan measure zero. This may seem to contradict the theorem above, but the error is trivial: we are not considering arbitrary sets (u, v), but those defined parametrically by functions $u(x, y)$, $v(x, y)$ satisfying (1.1). Such sets, when they are of Lebesgue measure zero, actually have the simple structure given in paragraph 2.

8. The above argument is suitable for the teaching of elementary analysis. But a respectable discussion of the question would consider conditions less restrictive than (1.1). We do not pretend to have done anything towards this.

I do not know if the exceptional set occurs (that is, cannot be lodged in an enumerable infinity of differentiable curves). G. K.

1726. The Government [of Venice in the 18th century] tried to keep up to date. The Arsenal, which was the naval office and dockyard, maintained a professor of pure mathematics and a professor of applied mathematics.—R. B. Mowat, *The Age of Reason*, p. 164.

1727. The mathematically-minded are the worldlings. Commerce is their natural and customary goal. In its turn it finds in them those alert, dexterous and precise qualities of mind which it needs, and which it is always ready to reward.—Letter in the *Daily Telegraph*, June 29, 1951.

1728. He [Bailly] was a mathematician and of a dreamy disposition—for mathematicians are poets—and he was consequently quite the opposite of a statesman.—Louis Madelin, *The Revolutionaries, 1789–1799*, p. 107.

1729. The problem of the scientist who wishes to perform a numerical calculation is to find a number.—L. M. Milne-Thomson, *Jacobian elliptic function tables* (Dover, 1951), p. ix. [Per Professor E. H. Neville.]

SOME REMARKS ON EQUILATERAL TRIANGLES AND SQUARES.

BY R. W. WEITZENBÖCK.

I. Triangles.

Let a_1, a_2, a_3 denote the sides $\alpha_1, \alpha_2, \alpha_3$ the angles and A_1, A_2, A_3 the vertices of a triangle in the Euclidean plane. A point P, whose distances from the sides a_1, a_2, a_3 are in the ratio $p_1 : p_2 : p_3$ will be denoted by $P[p_1]$. p_1, p_2, p_3 are called the normal coordinates of P. Thus the unit point $E[1]$ is the in-centre of the triangle, $S[\operatorname{cosec} \alpha_1]$ is the centroid, $M[\cos \alpha_1]$ is the circumcentre, and $H[\sec \alpha_1]$ is the orthocentre. Similarly, using normal line coordinates, we have $l_0[1]$ is the unit line, $l_\infty[\sin \alpha_1]$ is the line at infinity and $l_H[\cos \alpha_1]$ is the axis of the altitudes.

(*A*) *Circumscribed equilateral triangles Δ_c.*

It is easily proved that there exist two systems Σ_c and Σ_c', each of ∞^1 equilateral triangles $\triangle_c$ circumscribing $A_1A_2A_3$. Each of these systems contains a minimum triangle which coincides with the point $T\left[\dfrac{1}{\sin \alpha_1 + \sqrt{3} \cos \alpha_1}\right]$ (the point of Torricelli) in the case of Σ_c, or $T'\left[\dfrac{1}{\sin \alpha_1 - \sqrt{3} \cos \alpha_1}\right]$ in the case of Σ_c' and also a maximum triangle $\triangle_m$ or $\triangle_m'$. The lengths ρ_m and ρ_m' of the sides of $\triangle_m$ and $\triangle_m'$ are given by the formulae

$$\rho_m^2 = \tfrac{8}{3}F\sqrt{3} + \tfrac{2}{3}(a_1^2 + a_2^2 + a_3^2), \quad \rho_m'^2 = -\tfrac{8}{3}F\sqrt{3} + \tfrac{2}{3}(a_1^2 + a_2^2 + a_3^2),$$

where F is the area of $A_1A_2A_3$. The sides of $\triangle_m$ are perpendicular to the corresponding sides of $\triangle_m'$. Let M_0 and M_0' be the centres of the triangles $\triangle_m$ and $\triangle_m'$ respectively. Then the centroid S of $A_1A_2A_3$ bisects the segments M_0T and $M_0'T'$. These centres are therefore the points

$$M_0\left[\frac{\sin \alpha_1 + 2\sqrt{3} \cos \alpha_1}{\sin \alpha_1(\sin \alpha_1 + \sqrt{3} \cos \alpha_1)}\right], \quad M_m'\left[\frac{\sin \alpha_1 - 2\sqrt{3} \cos \alpha_1}{\sin \alpha_1(\sin \alpha_1 - \sqrt{3} \cos \alpha_1)}\right].$$

On the perpendicular bisector of A_2A_3 is a point N_1, inside $A_1A_2A_3$, and another point N_1' outside $A_1A_2A_3$, at which A_2A_3 subtends an angle of 120°. The triangles $N_1N_2N_3$ and $N_1'N_2'N_3'$ are equilateral and their centroid is S. The sides of $N_1N_2N_3$ are parallel to the sides of $\triangle_m$ and T is the centre of perspective of these triangles. Analogously T' is the centre of perspective of $N_1'N_2'N_3'$ and $\triangle'_m$.

The locus of the ∞^1 centres of the triangles $\triangle_c$ of Σ_c is a circle K_c whose centre is S and whose radius is $R_c = \dfrac{1}{2\sqrt{3}}\rho_m'$; TM_0 is a diameter of the circle K_c. Analogously for the system Σ_c' we find a circle K_c' of radius $R_c' = \dfrac{1}{2\sqrt{3}}\rho_m$.

(*B*) *Inscribed equilateral triangles $\triangle_i$.*

There are two systems Σ_i and Σ_i' each of ∞^1 equilateral triangles $\triangle_i$ and $\triangle_i'$ respectively, which can be inscribed in $A_1A_2A_3$. In each system there is an infinite triangle and a minimum triangle $\triangle_i(\triangle_i')$. The coordinates of P_i and P_i', given by the centres of $\triangle_i$ and $\triangle_i'$, are $[\sin \alpha_2 \sin \alpha_3 \pm (\sqrt{3}/2) \sin \alpha_1]$. These points are thus collinear with S and the Lemoine point $[\sin \alpha_1]$. The sides of these minimum triangles are of length $\sigma = 2F/\rho_m\sqrt{3}$, ρ_m having the same meaning as in (*A*).

The centres of $\triangle_i$ and $\triangle_i'$ lie on two parallel lines $L[\cos(\alpha_1 + 30°)]$ and $L'[\cos(\alpha_1 - 30°)]$, which are the harmonic polars of the isogonic points T and

T' with respect to the triangle $A_1A_2A_3$. Midway between these lines lies the axis of the altitudes [cos α_1]. The square of the distance between L and L' is $a_1^2a_2^2a_3^2/[12\{9R^2-(a_1^2+a_2^2+a_3^2)\}]$ where

$$R=\frac{a_1}{2\sin\alpha_1}=\frac{a_2}{2\sin\alpha_2}=\frac{a_3}{2\sin\alpha_3}=\text{radius of the circumcircle.}$$

The sides of the triangles $\triangle_i$ and $\triangle_i'$ envelop two systems of three parabolas.

II. *Rectangles and Squares.*

Four points $A(A_1, A_2), \ldots, D(D_1, D_2)$ in the xy-plane determine a quadrangle. The distances $s_{12}=\overline{AB}$, $s_{13}=\overline{AC}$, ... are supposed to be positive lengths defined by

$$s_{12}=\overline{AB}=|\sqrt{[(A_1-B_1)^2+(A_2-B_2)^2]}|=|\sqrt{[x_{12}^2+y_{12}^2]}|>0, \text{ etc.}$$

We suppose moreover that no three of the four points A, B, C and D are collinear.

We denote by $\alpha_{12,34}$ the angle measured in the positive sense between the directed lines AB and CD, so that

$$\cos\alpha_{12,34}=\frac{D_1-C_1}{s_{34}}\cdot\frac{B_1-A_1}{s_{12}}+\frac{D_2-C_2}{s_{34}}\cdot\frac{B_2-A_2}{s_{12}}=\frac{x_{12}x_{34}+y_{12}y_{34}}{s_{12}s_{34}},$$

$$\sin\alpha_{12,34}=\frac{D_2-C_2}{s_{34}}\cdot\frac{B_1-A_1}{s_{12}}-\frac{D_1-C_1}{s_{34}}\cdot\frac{B_2-A_2}{s_{12}}=\frac{x_{12}y_{34}-y_{12}x_{34}}{s_{12}s_{34}}.$$

(*A*) *Circumscribed rectangles.*

The distances of the points A and C from the line $y=\lambda x$ ($\lambda^2\neq-1$) are

$$\frac{\lambda A_1-A_2}{\sqrt{[1+\lambda^2]}} \quad\text{and}\quad \frac{\lambda C_1-C_2}{\sqrt{[1+\lambda^2]}}.$$

Their difference gives the breadth d_{13} of the " λ-strip " through A and C:

$$d_{13}=\frac{\lambda x_{13}-y_{13}}{\sqrt{[1+\lambda^2]}}. \quad\text{.........(1)}$$

Similarly, we obtain for the $(-1/\lambda)$-strip through B and D (perpendicular to the λ-strip)

$$d_{24}=\frac{x_{24}+\lambda y_{24}}{\sqrt{[1+\lambda^2]}}. \quad\text{.........(2)}$$

The mid-lines of these strips have therefore the equations

$$\left.\begin{aligned} y-\lambda x-\tfrac{1}{2}(A_2+C_2)+\tfrac{1}{2}\lambda(A_1+C_1)=0,\\ \lambda y+x-\tfrac{1}{2}\lambda(B_2+D_2)-\tfrac{1}{2}(B_1+C_1)=0;\end{aligned}\right\} \quad \begin{aligned}&\text{.........(3)}\\&\text{.........(4)}\end{aligned}$$

their point of intersection $M_{13,24}(\lambda)$ is given by

$$\left.\begin{aligned} x_m=x_m(\lambda)=[\lambda^2(A_2+C_2)-\lambda(A_2+C_2-B_2-D_2)+(B_1+D_1)]/2(1+\lambda^2),\\ y_m=y_m(\lambda)=[\lambda^2(B_2+D_2)-\lambda(A_1+C_1-B_1-D_1)+(A_2+C_2)]/2(1+\lambda^2).\end{aligned}\right\}\ldots(5)$$

From this it follows that

$$\tfrac{1}{2}[x_m(\lambda)+x_m(-1/\lambda)]=\tfrac{1}{4}(A_1+B_1+C_1+D_1),$$

that is to say, the centroid S of the quadrangle is the mid-point of the line

$$M_{13,24}(\lambda)M_{13,24}(-1/\lambda).$$

The same is true for $M_{12,34}$ and $M_{14,23}$.

The two strips (1) and (2) have the circumscribed rectangle $T_{13,24}(\lambda)$ in common; the point $M_{13,24}(\lambda)$ of (5) is therefore the mid-point of this rectangle. Eliminating λ we obtain from (5)

$$\begin{aligned}16(x^2+y^2)&[(A_1+C_1-B_1-D_1)^2+(A_2+C_2-B_2-D_2)^2]\\ &-8x(A_1+B_1+C_1+D_1)[\ldots]-8y(A_2+B_2+C_2+D_2)[\ldots]\\ &-\{[(A_1+C_2-B_1-D_1)^2-4(A_2+C_2)(B_2+D_2)](A_2+C_2-B_2-D_2)^2\\ &-4(A_1+C_1)(B_1+D_1)]-(A_1+B_1+C_1+D_1)^2(A_2+B_2+C_2+D_2)^2\}=0. \quad \ldots(6)\end{aligned}$$

Thus, the locus of the mid-points $M_{13,24}(\lambda)$ is a circle with S as centre and $R_{13,24}$ as radius, where

$$\left.\begin{aligned}16R_{13,24}&=(A_1+C_1-B_1-D_1)^2+(A_2+C_2-B_2-D_2)^2\\ &=(x_{12}+x_{34})^2+(y_{12}+y_{34})^2=s_{12}^2+s_{34}^2+2s_{12}s_{34}\cos\alpha_{12,34}\\ &=(x_{14}+x_{32})^2+(y_{14}+y_{32})^2=s_{14}^2+s_{23}^2-2s_{14}s_{23}\cos\alpha_{14,23}\end{aligned}\right\} \quad \ldots\ldots(7)$$

From these and the corresponding equations for $R_{12,34}^2$ and $R_{14,23}^2$, we find by eliminating the cosine terms

$$\left.\begin{aligned}8(R_{12,34}^2+R_{14,23}^2)&=s_{13}^2+s_{24}^2,\\ 8(R_{13,24}^2+R_{12,34}^2)&=s_{14}^2+s_{23}^2,\\ 8(R_{14,23}^2+R_{13,24}^2)&=s_{24}^2+s_{34}^2;\end{aligned}\right\} \quad \ldots\ldots(8)$$

and so finally

$$\left.\begin{aligned}16R_{12,34}^2&=-s_{12}^2-s_{34}^2+s_{13}^2+s_{24}^2+s_{14}^2+s_{23}^2,\\ 16R_{13,24}^2&=s_{12}^2+s_{34}^2-s_{13}^2-s_{24}^2+s_{14}^2+s_{23}^2,\\ 16R_{14,23}^2&=s_{12}^2+s_{34}^2+s_{13}^2+s_{24}^2-s_{14}^2-s_{23}^2.\end{aligned}\right\} \quad \ldots\ldots(9)$$

We therefore have the following theorem: *There are three systems of* ∞^1 *rectangles which circumscribe a given quadrangle ABCD. The mid-points of the rectangles of each system lie on a circle whose centre is the centroid of the quadrangle ABCD and whose radius is given by* (9).

(*B*) *Circumscribed squares.*

It follows from (1) and (2) that the rectangle $T_{13,24}(\lambda)$ is a square $Q_{13,24}$, if and only if λ satisfies one of the equations $\lambda x_{13}-y_{13}=\pm(x_{24}+\lambda y_{24})$, that is to say, if λ takes one of the values

$$\lambda_{13,24}=\frac{x_{24}+y_{13}}{x_{13}-y_{24}}, \quad \lambda_{13,24}=\frac{y_{13}+x_{24}}{x_{13}+y_{24}}. \quad \ldots\ldots(10)$$

Substituting these values in $(\lambda x_{13}-y_{13})^2/(1+\lambda^2)$, we determine the lengths q' and q'' of the sides of the two squares $Q_{13,24}$:

$$\left.\begin{aligned}q'^2_{13,24}&=\frac{s_{13}^2s_{24}^2\cos^2\alpha_{13,24}}{s_{13}^2+s_{24}^2-2s_{13}s_{24}\sin\alpha_{13,24}},\\ q''^2_{13,24}&=\frac{s_{13}^2s_{24}^2\cos^2\alpha_{13,24}}{s_{13}^2+s_{24}^2+2s_{13}s_{24}\sin\alpha_{13,24}}.\end{aligned}\right\} \quad \ldots\ldots(11)$$

Now, it is easily calculated that

$$s_{13}s_{24}\cos\alpha_{13,24}=\tfrac{1}{2}(s_{14}^2+s_{23}^2-s_{12}^2-s_{34}^2), \quad \ldots\ldots(12)$$

and thus $\quad s_{13}^2s_{24}^2\sin^2\alpha_{13,24}=\tfrac{1}{4}[4s_{13}^2s_{24}^2-(s_{14}^2+s_{23}^2-s_{12}^2-s_{34}^2)^2]. \quad \ldots\ldots(13)$

Consequently, (11) takes the form

$$\left.\begin{aligned}q'^2_{13,24}&=\tfrac{1}{4}\,\frac{(s_{24}^2+s_{23}^2-s_{12}^2-s_{3}^2\ )^2}{s_{13}^2+s_{24}^2-\sqrt{(4s_{13}^2s_{24}^2-(s_{14}^2+s_{23}^2-s_{12}^2-s_{34}^2)^2)}},\\ q''^2_{13,24}&=\tfrac{1}{4}\,\frac{(s_{14}^2+s_{23}^2-s_{12}^2-s_{34}^2)^2}{s_{13}^2+s_{24}^2+\sqrt{(4s_{13}^2s_{24}^2-(s_{14}^2+s_{23}^2-s_{12}^2-s_{34}^2)^2)}}.\end{aligned}\right\} \quad \ldots\ldots(14)$$

In general therefore we obtain six different squares circumscribed by a given quadrangle $ABCD$. The sides $q'_{13,24}$ and $q'_{14,23}$ can be obtained from (14) by permutations of the points A, B, C, and D. To construct $Q'_{13,24}$ and $Q''_{13,24}$ we draw through B a line perpendicular to AC and construct on this line two points B' and B'' such that $BB'=BB''=s_{13}$. Then, one side of the square $Q_{13,24}$ lies on the line $B'D$; its opposite side is to be found on the parallel to $B'D$ through the point B. In the same way $B''D$ and the square $C_{13,24}$ may be constructed. It should be remarked that in general the squares $Q_{13,24}$ do not coincide with the circumscribed rectangles $T_{13,24}$ of extreme area. These areas are given by

$$(T_{13,24})\text{extr.} = \tfrac{1}{2}s_{13}s_{24}(1 \pm \sin \alpha_{13,24}). \qquad \text{......................(15)}$$

(*C*) *Inscribed rectangles.*

To investigate the rectangles and squares inscribed in a quadrangle, it is more convenient to start from four lines a, b, c and d, whose equations are

$$a_x = a_1x + a_2y + a_0 = 0, \; \ldots \;, d_x = d_1x + d_2y + d_0 = 0.$$

It is supposed that no three of them are concurrent and that, moreover, no line is an isotropic one, that is to say, the Euclidean invariants $(a \mid a) = a_1^2 + a_2^2$, etc. do not vanish. $(a \mid b) = a_1b_1 + a_2b_2 = 0$ expresses the orthogonality of a and b.

The four corners of a rectangle with centre $P_0(x_0.\, y_0)$ may be supposed to be of the form

$$P_1(x_0 + r\alpha', y_0 + r\beta'), \quad P_2(x_0 + r\alpha'', y_0 + r\beta''),$$
$$P_3(x_0 - r\alpha', y_0 - r\beta'), \quad P_4(x_0 - r\alpha'', y_0 - r\beta'').$$

Here α', β' ; α'', β'' are the direction cosines and $2r$ is the length of the diagonals.

The conditions that these four corners should lie on the lines a, b, c and d are expressed by the equations

$$\left.\begin{aligned} a_x + r(\;\; a_1\alpha' + a_2\beta') &= 0,\\ b_x + r(\;\; b_1\alpha'' + b_2\beta'') &= 0,\\ c_x + r(-c_1\alpha' - c_2\beta') &= 0,\\ d_x + r(-d_1\alpha'' - d_2\beta'') &= 0. \end{aligned}\right\} \qquad \text{........................(16)}$$

These are four equations with five unknowns x_0, y_0, r, α'/β', and α''/β'' ; so we will have in general ∞^1 rectangles $T_i(abcd)$.

If $(ac) \equiv (ac)_{12} \equiv a_1c_2 - a_2c_1$ vanishes, then the lines a and c are parallel. If, however, $(ac) \neq 0$ and $(bd) \neq 0$, we find from (16) that

$$\left.\begin{aligned} r\alpha' &= \frac{c_2a_x + a_2c_x}{(ac)_{12}}, \quad r\beta' = -\frac{c_1a_x + a_1c_x}{(ac)_{12}},\\ r\alpha'' &= \frac{d_2c_x + c_2d_x}{(cd)_{12}}, \quad r\beta'' = -\frac{d_1c_x + c_1d_x}{(cd)_{12}}. \end{aligned}\right\} \qquad \text{..................(17)}$$

Since $\alpha'^2 + \beta'^2 = \alpha''^2 + \beta''^2 = 1$, we get

$$\left.\begin{aligned} r^2 &= \frac{1}{(ac)^2}\,[a_x^2(c \mid c) + 2a_xc_x(c \mid a) + c_x^2(a \mid a)]\\ &= \frac{1}{(cd)^2}\,[b_x^2(d \mid d) + 2b_xd_x(b \mid b) + d_x^2(b \mid b)]. \end{aligned}\right\} \qquad \text{..................(18)}$$

Thus the locus of the centre P_0 of the inscribed rectangles $T_i(abcd)$ is a conic. If $(ac) = 0$, we may put $c_x = a_x$; then from (16) we conclude that $2a_{x_0} = 0$, since $a_{x_0} + c_{x_0} = 0$. In this case a and c are the two parallel sides of a trapezium

and P_0 lies on the line joining the mid-points of these sides. If also $(bd)=0$, then $abcd$ is a parallelogram and all centres P_0 coincide with its centroid. These facts are geometrically evident.

(D) *The inscribed squares* Q_i.

The four corners of a square, whose diagonals are of length $2r$ and whose centre is P_0, can be represented by

$$\left.\begin{aligned} &P_1(x_0+r\alpha,\ y_0+r\beta), \quad P_2(x_0+r\beta,\ y_0+r\alpha),\\ &P_3(x_0-r\alpha,\ y_0-r\beta), \quad P_4(x_0+r\beta,\ y_0-r\alpha).\end{aligned}\right\} \qquad \text{(19)}$$

These points lie on the sides a, b, c and d if

$$\left.\begin{aligned} a_1x_0+a_2y_0+a_1\,.\,r\alpha+a_2\,.\,r\beta+a_0&=0,\\ b_1x_0+b_2y_0+b_2\,.\,r\alpha-b_1\,.\,r\beta+b_0&=0,\\ c_1x_0+c_2y_0-c_1\,.\,r\alpha-c_2\,.\,r\beta+c_0&=0,\\ d_1x_0+d_2y_0-d_2\,.\,r\alpha+d_1\,.\,r\beta+d_0&=0.\end{aligned}\right\} \qquad \text{(20)}$$

These are four linear equations with four unknowns x_0, y_0, $r\alpha$ and $r\beta$; so we will have generally exactly one square $Q_i(abcd)$, where P_1 lies on a, P_2 on b, P_3 on c and the point P_4 on the line d. We emphasise that, in the following, the order of succession of the four points P given by (19) is not changed. Then the square $Q_i(pqrs)$ will depend on the permutation $pqrs$ of $abcd$.

In the further discussion of the different possibilities which may occur the matrix

$$D_{12340}=\left\| \begin{matrix} a_1 & a_2 & a_1 & a_2 & a_0\\ b_1 & b_2 & b_2 & -b_1 & b_0\\ c_1 & c_2 & -c_1 & -c_2 & c_0\\ d_1 & d_2 & -d_2 & d_1 & d_0 \end{matrix}\right\| \qquad \text{(21)}$$

of the coefficients from (20), and in particular the first determinant of this matrix

$$D_{1234}=\left| \begin{matrix} a_1 & a_2 & a_1 & a_2\\ b_1 & b_2 & b_2 & -b_1\\ c_1 & c_2 & -c_1 & -c_2\\ d_1 & d_2 & -d_2 & d_1 \end{matrix}\right| \qquad \text{(22)}$$

will play a decisive and important role. We have

$$\begin{aligned} \tfrac{1}{2}D_{1234}&=(ac)(bd)-(ab)(c\mid d)+(ac)(b\mid d)-(ad)(b\mid c)\\ &=p_1p_2p_3p_4[\sin(a,c)\sin(b,d)-\sin(a,b)\cos(c,d)\\ &\qquad +\sin(a,c)\cos(b,d)-\sin(a,d)\cos(b,c)], \end{aligned} \qquad \text{(23)}$$

where p_1 is the distance of the line a from the origin and the angles (a, c) etc. are measured in the positive sense of rotation.

We suppress here a detailed discussion of the possible solutions of the equations (20) and give only those results most worthy of remark.

In the first place, the rank h of the matrix (21) is at least 3. If $h=4$ and $D_{1234}=0$, then $Q_i(abcd)$ does not exist. Secondly, if $h=3$ and $D_{1234}=0$, we find in general ∞^1 inscribed squares $Q_i(abcd)$, whose centres lie on a straight line m. Here we have two exceptional cases. In the first of these m is the line at infinity and there is no proper square $Q_i(abcd)$. In this case a is perpendicular to c and b is perpendicular to the line d; moreover, the angle between a and b, and hence also the angles (a, d) and (c, d) are 45°.

The second exception occurs if there is no line m but ∞^1 squares $Q_i(abcd)$ with the same centre P_0; it can be shown that in this case $abcd$ itself is a square with centre P_0.

(*E*) *The different $Q_i(pqrs)$.*

Let *pqrs* denote a permutation of *abcd*. Because of the binary identities

$$(ab)(cd)+(bc)(ad)+(ca)(bd)=0,$$
$$(ab)(c \mid d)+(bc)(a \mid d)+(ca)(b \mid d)=0,$$

we derive from (23) the following relations:

$$D_{1234}=-D_{2341}=+D_{3412}=-D_{4123}, \quad\ldots\ldots(24)$$
$$D_{3214}=D_{1234}-4(ac)(bd), \quad\ldots\ldots(25)$$
$$D_{2134}=D_{1234}-2(ab)(cd)+4(ab)(c \mid d), \quad\ldots\ldots(26)$$
$$D_{1234}+D_{1243}+D_{1342}+D_{1432}=-8(ab)(c \mid d), \quad\ldots\ldots(27)$$
$$D_{1234}-D_{1243}+D_{1342}-D_{1324}+D_{1423}-D_{1432}=0. \quad\ldots\ldots(28)$$

(24) shows that we may confine our attention to six of the twenty-four D_{pqrr} only, *e.g.*, those with the first index 1. From (25) and (28) we learn that at most four of the six D_{1qrs} may vanish; hence, there exist at least two inscribed squares $Q_i(pqrs)$. Furthermore, it is easily demonstrated that there exists at most one system of ∞^1 squares.

In the general case in which all six D_{1qrs} are non-zero, we have six inscribed squares Q_i. From the equations (20) the lengths σ_{1qrs} of their sides are easily calculated. We have, *e.g.*,

$$D^2_{1234}\sigma^2_{1234}=\Sigma(a \mid a)A^2+2(ab)AB-2(a \mid c)AC-2(ad)AD$$
$$+2(bc)BC-2(b \mid d)BD+2(cd)CD\ldots\ldots(29)$$

where

$$A=(bcd)=\begin{vmatrix} b_1 & b_2 & b_0 \\ c_1 & c_2 & c_0 \\ d_1 & d_2 & d_0 \end{vmatrix}, \quad B=-(acd), \quad C=(abd), \quad D=-(abc).$$

(*F*) *Construction of the inscribed squares Q_i.*

First step: from a point C arbitrarily chosen on the line c, we draw CC' perpendicular to a, C' lying on a. Through the mid-point M' of the segment CC' we draw a line a' parallel to a. Choose on this line a' an arbitrary point M and construct the square with centre M and MC as half its diagonal. Let $ABCD$ be this square, the corner A lying on a. Now it is easy to show that the points D and B move along lines l_D and l_B if M moves along the line a'; the point C, however, remains fixed. The lines l_B and l_D are the bisectors of the right angles at C'. If B is the point of intersection of l_B and b, then $ABCD$ is a square with A on a, B on b, and C on c. The problem would now be solved provided the point D also lies on d.

Second step: To each point C on c we will find in this way a point D. It can easily be proved that the points $D_1, D_2, \ldots$, which correspond to different points $C_1, C_2, \ldots$ on c are situated on a straight line m. We draw this line by executing twice the construction described in the first step. The line m meets d in a point D_0 and by retracing the steps of the construction we find the required square $A_0B_0C_0D_0=Q_i(abcd)$. By interchanging the lines l_B and l_D we get in the same manner $Q_i(adcb)$. R. W. W.

1730. [The term] *Kinetic energy* was first used in an article by W. Thomson and P. G. Tait in *Good Words* (then edited by Charles Dickens) October 1862.—Sir Edmund Whittaker, *History of the theories of Aether and Electricity*, I (1951), p. 214.

1731. . . . Standard theorems on the continuation of integral curves are reproved.—Report in *Mathematical Reviews*, XII, No. 8, p. 611. [Per Professor L. M. Milne-Thomson.]

THE RANK AND MULTIPLICITY THEOREM FOR THE REDUCTION OF QUADRATIC FORMS.

By S. N. Afriat.

1. *Introduction.* Direct and independent proof is given to the theorem, implicit in the possibility, and fundamental for the procedure of the simultaneous canonical reduction of a pair of real quadratic forms, one of which is positive definite. In this way a complete unification of the possibility and the procedure is effected.

Recently Ferrar (1) has considered this question of reduction by a method "free of difficult invariant factor arguments", and Todd (2) the example of the orthogonal reduction of a single form in a direct and non-inductive fashion to remove, with reference to the traditional method for proving the possibility, "the aesthetic objection that it does not correspond to the practical method for obtaining the reduction". It is from the latter that suggestion has been discovered for the procedure given here by which the objects of both authors are achieved in combination.

Proof is also given to the linear independence question referred to by Todd in a manner which he suggests has been lacking.

2. The following simple lemmas are required.

Lemma I. $$|\mathbf{a}+\mathbf{b}| = \sum_{\nu=0}^{n} \text{trace } \mathbf{a}^{(\nu)}\mathbf{b}^{[\nu]}. \quad (3)$$

For the coefficient of $a^{(\nu)}_{ij}$ in $|\mathbf{a}+\mathbf{b}|$ is the coefficient of $b^{(\nu)}_{ij}$ in $|\mathbf{b}|$, that is, $b^{[\nu]}_{ji}$.

Lemma II. *If* $\mathbf{a}$ *is of rank* r *then* $\mathbf{a}^{(r)}$ *is of rank* 1. (2)

For then $\mathbf{a}=\boldsymbol{\theta}\boldsymbol{\varphi}'$, where $\boldsymbol{\theta}$, $\boldsymbol{\varphi}$ have r independent columns, so that $\mathbf{a}^{(r)}=\boldsymbol{\theta}^{(r)}\boldsymbol{\varphi}'^{(r)}$ is of rank 1.

Lemma III. *If* $\mathbf{a}$ *is positive definite then so are all its adjugates* $\mathbf{a}^{[k]}$ (4).

For the characteristic values of $\mathbf{a}^{[k]}$ are products of characteristic values of $\mathbf{a}$ and hence are positive.

Lemma IV. *If* $\mathbf{a}$ *is real, symmetric, and of rank* 1, *then* $\mathbf{a}=\epsilon\boldsymbol{\theta}\boldsymbol{\theta}'$ *where* $\boldsymbol{\theta}$ *is a real column vector and* $\epsilon=\pm 1$.

For then $aa_{kkij}=a_{ik}a_{kj}=a_{ki}a_{kj}$; and some $a_{kk}\neq 0$ since $\mathbf{a}\neq\mathbf{0}$. Thus $a_{ij}=a_{ki}a_{kj}a_{kk}{}^{-1}=\epsilon\theta_i\theta_j$, where $\theta_i=a_{ki}\,|\,\epsilon a_{kk}\,|^{-\frac{1}{2}}$ and $\epsilon=a_{kk}\,|\,a_{kk}\,|^{-1}$.

3. Theorem. *If* $\mathbf{a}$, $\mathbf{b}$ *is a pair of real symmetric matrices of which* $\mathbf{b}$ *is positive definite, and if* λ_0 *is a root of multiplicity* m_0 *of the equation* $|\mathbf{a}-\lambda\mathbf{b}|=0$, *then the matrix* $\mathbf{a}-\lambda_0\mathbf{b}$ *has nullity* m_0.

For then, by Lemma I, $\mu=0$ will be an m_0-fold root of the equation

$$\sum_{r=0}^{n}(-\mu)^{n-r}\text{ trace }\{\mathbf{c}_0^{(r)}\mathbf{b}^{[r]}\}=0,$$

where $\mathbf{c}_0=\mathbf{a}-\lambda_0\mathbf{b}$ is symmetric, and real since λ_0 is real. Thus

$$\text{trace }\{\mathbf{c}_0^{(r)}\mathbf{b}^{[r]}\}=0, \quad r_0<r\leqslant n,$$

$$\text{trace }\{\mathbf{c}_0^{(r_0)}\mathbf{b}^{[r_0]}\}\neq 0,$$

where $r_0=n-m_0$. If $\mathbf{c}_0$ is of rank ρ, then $\mathbf{c}_0^{(\rho)}=\epsilon\boldsymbol{\theta}\boldsymbol{\theta}'$, by lemmas II and IV. Further, $r_0\leqslant\rho$, since $\mathbf{c}_0^{(r_0)}\neq\mathbf{0}$; and the hypothesis $r_0>\rho$ gives

$$0=\text{trace }\{\mathbf{c}_0^{(\rho)}\mathbf{b}^{[\rho]}\}=\epsilon\boldsymbol{\theta}'\mathbf{b}^{[\rho]}\boldsymbol{\theta},$$

which implies $\boldsymbol{\theta}=\mathbf{0}$, by lemma III, which makes a contradiction. Hence we have $r_0=\rho$ as required.

4. *The Reduction.* Let $\lambda_1, \ldots, \lambda_n$ be the roots of the equation $|\,\mathbf{a}-\lambda\mathbf{b}\,|=0$ in natural order, all being real; and let $\mu_1, \ldots, \mu_s$, $s\leqslant n$, be the distinct roots in the same order, μ_k having multiplicity m^k, so that $m_1+\ldots+m_s=n$. Let $\mathfrak{X}_k$ be the null space of $\mathbf{a}-\mu_k\mathbf{b}$; which, by the rank and multiplicity theorem, is spanned by some m_k independent vectors forming the columns of a matrix $\mathbf{P}_k$, of full rank, such that $\mathbf{aP}_k=\mu_k\mathbf{bP}_k$. Then the matrix $\mathbf{P}=(\mathbf{P}_1\ldots\mathbf{P}_s)$ is regular. For if there existed a relation of linear dependence between its columns, it could be written in the form $\mathbf{X}\equiv\mathbf{X}_1+\ldots+\mathbf{X}_s=\mathbf{0}$, where the vector $\mathbf{X}_k$ belongs to the space $\mathfrak{X}_k$. Then it would follow that

$$\mathbf{a}^r\mathbf{X}=\mathbf{b}^r(\mu_1{}^r\mathbf{X}_1+\ldots+\mu_s{}^r\mathbf{X}_s)=0, \qquad 0\leqslant r\leqslant s-1,$$

and then that

$$\mu_1{}^r\mathbf{X}_1+\ldots+\mu_s{}^r\mathbf{X}_s=0, \qquad 0\leqslant r\leqslant s-1,$$

since $\mathbf{b}^k$ is regular with $\mathbf{b}$; and this implies the vanishing of the simple alternant $|\,\mu_1{}^0\ldots\mu_s{}^{s-1}\,|$, which would contradict the distinctness of the μ_k. For the elements of the matrices $\boldsymbol{\alpha}=\mathbf{P}'\mathbf{aP}$, $\boldsymbol{\beta}=\mathbf{P}'\mathbf{bP}$, we now have

$$\alpha_{ij}=\lambda_i\beta_{ij}=\lambda_j\beta_{ij},$$

by symmetry; so that $\alpha_{ij}=\beta_{ij}=0$ when $\lambda_i\neq\lambda_j$. Thus

$$\mathbf{P}_h{}'\mathbf{aP}_k=\mathbf{0}, \qquad \mathbf{P}_h{}'\mathbf{bP}_k=\mathbf{0}, \qquad (h\neq k);$$

and if $\boldsymbol{\alpha}_k=\mathbf{P}_k{}'a\mathbf{P}_k$, $\boldsymbol{\beta}_k=\mathbf{P}_k{}'b\mathbf{P}_k$, we have

$$\boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\alpha}_1 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \boldsymbol{\alpha}_s\end{pmatrix}, \quad \boldsymbol{\beta}=\begin{pmatrix}\boldsymbol{\beta}_1 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \boldsymbol{\beta}_s\end{pmatrix},$$

where $\boldsymbol{\alpha}_k=\mu_k\boldsymbol{\beta}_k$, or $\boldsymbol{\alpha}=\boldsymbol{\lambda\beta}$, where

$$\boldsymbol{\lambda}=\begin{pmatrix}\mu_1\mathbf{1}_{m_1} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \mu_s\mathbf{1}_{m_s}\end{pmatrix}=\begin{pmatrix}\lambda_1 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \lambda_n\end{pmatrix}.$$

Thus when the $\lambda_1, \ldots, \lambda_n$ are distinct the reduction is all but complete; in any case, the $\boldsymbol{\beta}_k$ being positive definite with $\mathbf{b}$, there are, by repeated congruent isolation of diagonal elements, regular matrices $\mathbf{Q}_k$ such that $\mathbf{Q}_k{}'\boldsymbol{\beta}_k\mathbf{Q}_k=\mathbf{I}_{mk}$; so that the regular matrix $\mathbf{T}=\mathbf{PQ}$, where

$$\mathbf{Q}=\begin{pmatrix}\mathbf{Q}_1 & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_s\end{pmatrix}$$

is such that $\mathbf{T}'\mathbf{aT}=\boldsymbol{\lambda}$, $\mathbf{T}'\mathbf{bT}=\mathbf{I}$; and the reduction is now complete.

5. *References.*

(1) Ferrar, " The Simultaneous Reduction of Two Real Quadratic Forms," *Quart. J. Math. Oxford*, Ser. II, 186–92 (1947).

(2) Todd, " A Note on Real Quadratic Forms," *Quart. J. Math. Oxford*, Ser. II, 183–5 (1947).

(3) Aitken, *Determinants and Matrices* (Oliver & Boyd, London, 1942), pp. 68, 89, 102.

(4) Turnbull and Aitken, *Introduction to the Theory of Canonical Matrices* (London and Glasgow, 1932), p. 91.

S. N. A.

THE DIRECT PRODUCT OF MATRICES.

By E. H. Lloyd.

I. INTRODUCTION.

The concept of the direct product is widely used in the theory of groups, algebras, vector spaces and allied topics, but its use as a manipulative tool in matrix algebra is less common.

The present note shows how the direct product of matrices arises in elementary work, derives some of its more important properties, and gives some examples of its use.

II. THE DIRECT PRODUCT.

We may define the direct product of matrices as follows. Let $\mathbf{a}$, $\mathbf{b}$ be matrices which transform the vectors $\mathbf{x}$, $\mathbf{y}$ respectively; then the direct product $\mathbf{a} \times \mathbf{b}$ is the matrix which transforms the appropriately ordered set of binary products $x_i y_j$. Now, if $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are the vectors

$$\boldsymbol{\xi} = \mathbf{a}\mathbf{x}, \quad \boldsymbol{\eta} = \mathbf{b}\mathbf{y},$$

then

$$\xi_i \eta_j = \sum_{r,s} a_{ir} b_{js} x_r y_s, \quad \text{.................................(2.1)}$$

so that the $x_r y_s$ are transformed by a matrix whose typical element is $a_{ir} b_{js}$; the row being defined by the indices (i, j), and the column by (r, s). This matrix is the direct product $\mathbf{a} \times \mathbf{b}$. A convenient ordering convention is the "lexical" order.

$$11, 12, 13, \ldots, 21, 22, 23, \ldots, 31, 32, 33, \ldots.$$

With this convention the matrix $[a_{ir} b_{js}]$—that is to say, the matrix $\mathbf{a} \times \mathbf{b}$—may be written, in partitioned form, as follows

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_{11}\mathbf{b} & a_{12}\mathbf{b} & \ldots & a_{1\alpha_2}\mathbf{b} \\ a_{21}\mathbf{b} & a_{22}\mathbf{b} & \ldots & a_{2\alpha_2}\mathbf{b} \\ \ldots & \ldots & \ldots & \ldots \\ a_{\alpha_1 1}\mathbf{b} & \ldots & & a_{\alpha_1\alpha_2}\mathbf{b} \end{pmatrix}, \quad \text{..................(2.2)}$$

where $\mathbf{a}$ is of size $(\alpha_1 \times \alpha_2)$, and $\mathbf{b}$ of size $(\beta_1 \times \beta_2)$.

We may represent this by the notation

$$(\mathbf{a} \times \mathbf{b})_{ij} = a_{ij}\mathbf{b} \text{.....................................(2.3)}$$

for the (i, j)-th submatrix of $\mathbf{a} \times \mathbf{b}$. We note that the size of $\mathbf{a} \times \mathbf{b}$ is

$$(\alpha_1 \beta_1 \times \alpha_2 \beta_2).$$

With the same ordering convention, the vector of the binary products $x_r y_s$ in (2.1) is the direct product $\mathbf{x} \times \mathbf{y}$. Thus, using (2.1), we see that, if

$$\boldsymbol{\zeta} = \mathbf{a}\mathbf{x}, \quad \boldsymbol{\eta} = \mathbf{b}\mathbf{y},$$

then

$$\boldsymbol{\xi} \times \boldsymbol{\eta} = (\mathbf{a} \times \mathbf{b})(\mathbf{x} \times \mathbf{y}).$$

As an important example we note that if $\mathbf{I}_p$, $\mathbf{I}_q$ are the unit matrices of order p, q respectively, then

$$\mathbf{I}_p \times \mathbf{I}_q = \mathbf{I}_{pq}. \quad \text{....................................(2.4)}$$

As further examples we note that the direct product of column vectors is a column vector, the direct product of diagonal matrices is diagonal, the direct product of upper (lower) triangular matrices is an upper (lower) triangular matrix, and the direct product of scalars (regarded as (1×1) matrices) is equal to their ordinary product.

III. BASIC PROPERTIES.

The simpler basic properties of the direct product are as follows.

(i) *Associative and distributive properties :*

$$(a) \quad (\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times\mathbf{b}\times\mathbf{c}.$$

$$(b) \quad (\mathbf{a}+\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})\ ;$$
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c}).$$

(ii) *Multiplication by scalar :*

$$(h\mathbf{a})\times(k\mathbf{b})=hk(\mathbf{a}\times\mathbf{b}).$$

(iii) *Transposition :*

The transpose of a direct product is the direct product of the transposed matrices, in the same order.

$$(\mathbf{a}\times\mathbf{b})'=\mathbf{a}'\times\mathbf{b}'. \qquad \ldots\ldots(3.1)$$

(iv) *Product rule :*

The product rule for direct products is as follows :

$$(\mathbf{a}\times\mathbf{b})(\mathbf{c}\times\mathbf{d})=\mathbf{ac}\times\mathbf{bd}, \qquad \ldots\ldots(3.2)$$

subject to the usual conformability requirements for the ordinary product. (Note that there are no conformability requirements on the direct product; the direct product of two matrices always exists, whatever their sizes.) The product rule is proved as follows

$$\begin{aligned}\{(\mathbf{a}\times\mathbf{b})(\mathbf{c}\times\mathbf{d})\}_{ij}&=\sum_r(\mathbf{a}\times\mathbf{b})_{ir}(\mathbf{c}\times\mathbf{d})_{rj}\\&=\sum_r(a_{ir}\mathbf{b})(c_{rj}\mathbf{d})\\&=(\sum_r a_{ir}c_{rj})\mathbf{bd}=(\mathbf{ac})_{ij}\mathbf{bd}\\&=(\mathbf{ac}\times\mathbf{bd})_{ij}.\end{aligned}$$

Two obvious generalisations of the above rule are

$$(a) \quad (\mathbf{a}_1\times\mathbf{a}_2\times\ldots)(\mathbf{b}_1\times\mathbf{b}_2\times\ldots)=\mathbf{a}_1\mathbf{b}_1\times\mathbf{a}_2\mathbf{b}_2\times\ldots\ ;$$

$$(b) \quad (\mathbf{a}_1\times\mathbf{b}_1)(\mathbf{a}_2\times\mathbf{b}_2)(\mathbf{a}_3\times\mathbf{b}_3)\ldots=(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\ldots)\times(\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\ldots).$$

(v) *Inversion :*

An important corollary of the above rule (3.2) gives the inverse (when it exists) of a direct product :

$$(\mathbf{a}\times\mathbf{b})^{-1}=\mathbf{a}^{-1}\times\mathbf{b}^{-1}, \qquad \ldots\ldots(3.3)$$

provided $\mathbf{a}$ and $\mathbf{b}$ are both non-singular ; for, by (3.2) and (2.4),

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})(\mathbf{a}^{-1}\times\mathbf{b}^{-1})&=\mathbf{aa}^{-1}\times\mathbf{bb}^{-1}\\&=\mathbf{I}\times\mathbf{I}=\mathbf{I}.\end{aligned}$$

Similarly $\quad (\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\ldots)^{-1}=\mathbf{a}^{-1}\times\mathbf{b}^{-1}\times\mathbf{c}^{-1}\times\ldots.$

(vi) *Orthogonality :*

The direct product of orthogonal matrices is orthogonal ; for if $\mathbf{a}^{-1}=\mathbf{a}'$ and $\mathbf{b}^{-1}=\mathbf{b}'$, then by (3.1).

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})^{-1}&=\mathbf{a}^{-1}\times\mathbf{b}^{-1}=\mathbf{a}'\times\mathbf{b}'\\&=(\mathbf{a}\times\mathbf{b})'.\end{aligned}$$

This incidentally offers a useful method of constructing orthogonal matrices from smaller ones.

(vii) *Latent roots :*

The latent roots of a direct product can be elegantly expressed in terms of

those of its constituent matrices. They are in fact the set of all binary products which can be formed from the two separate sets of latent roots.

For, let **a**, **b** be square matrices with latent roots $\{\lambda_i\}$, $\{\mu_i\}$, respectively, not necessarily all distinct. There exists a non-singular matrix **L** such that

$$\mathbf{LaL}^{-1}=\boldsymbol{\lambda}^{\Delta},$$

where $\boldsymbol{\lambda}^{\Delta}$ is an upper triangular matrix whose diagonal elements are the latent roots λ_i of **a**. (This is the Jacobi cononical form). Likewise there is a non-singular matrix **M** such that

$$\mathbf{MbM}^{-1}=\boldsymbol{\mu}^{\Delta},$$

where $\boldsymbol{\mu}^{\Delta}$ is the (upper triangular) Jacobi canonical form of **b**. Then

$$\begin{aligned}\boldsymbol{\lambda}^{\Delta}\times\boldsymbol{\mu}^{\Delta}&=(\mathbf{LaL}^{-1})\times(\mathbf{MbM}^{-1})\\&=(\mathbf{L}\times\mathbf{M})(\mathbf{a}\times\mathbf{b})(\mathbf{L}^{-1}\times\mathbf{M}^{-1})\\&=(\mathbf{L}\times\mathbf{M})(\mathbf{a}\times\mathbf{b})(\mathbf{L}\times\mathbf{M})^{-1}.\end{aligned}$$

Since this matrix is also upper triangular, it follows from this equation that $\boldsymbol{\lambda}^{\Delta}\times\boldsymbol{\mu}^{\Delta}$ is the Jacobi canonical form of $\mathbf{a}\times\mathbf{b}$. Thus the latent roots of $\mathbf{a}\times\mathbf{b}$ are the diagonal elements of $\boldsymbol{\lambda}^{\Delta}\times\boldsymbol{\mu}^{\Delta}$, and these are all the binary products $\lambda_i\mu_j$, where the λ_i are the diagonal elements of $\boldsymbol{\lambda}^{\Delta}$ (latent roots of **a**), and the μ_j are the diagonal elements of $\boldsymbol{\mu}^{\Delta}$ (latent roots of **b**).

(viii) *Trace :*

The trace tr(**a**) of a square matrix **a** is the sum of its diagonal elements, and this equals the sum of the latent roots. It follows immediately from the latent roots theorem of § (vii) that the trace of a direct product equals the product of the separate traces :

$$\operatorname{tr}(\mathbf{a}\times\mathbf{b})=\operatorname{tr}(\mathbf{a})\,.\,\operatorname{tr}(\mathbf{b}).$$

Alternatively, this can be proved by the more elementary method :

$$\operatorname{tr}(\mathbf{a}\times\mathbf{b})=\sum_i(\mathbf{a}\times\mathbf{b})_{ii}$$

(subscripts here referring to *elements*, not submatrices)

$$=\Sigma\, a_{ii}\operatorname{tr}(\mathbf{b})=\operatorname{tr}(\mathbf{a})\,.\,\operatorname{tr}(\mathbf{b}).$$

(ix) *The determinant of a direct product :*

Since the determinant $|\,\mathbf{a}\,|$ of a square matrix **a** equals the product of it latent roots, it follows from § (vii) that, if **a** is $(p\times p)$ and **b** is $(q\times q)$, then

$$|\,\mathbf{a}\times\mathbf{b}\,|=|\,\mathbf{a}\,|^q\,|\,\mathbf{b}\,|^p. \qquad \text{.............................(3.4)}$$

This can also be proved by methods not invoking latent roots. For any determinant $|\,\mathbf{a}\,|$ may be reduced by elementary row- and column-operations to a diagonal form, say

$$|\,\mathbf{a}\,|=\begin{vmatrix}\lambda_1 & & \mathbf{0}\\ & \lambda_2 & \\ & & \ddots \\ \mathbf{0} & & \lambda_p\end{vmatrix}=\prod_{i=1}^{p}\lambda_i.$$

Applying these same operations to the "rows" and "columns" of submatrices $a_{ij}\mathbf{b}$ in the partitioned form of $\mathbf{a}\times\mathbf{b}$, we have

$$\begin{aligned}|\,\mathbf{a}\times\mathbf{b}\,|&=\begin{vmatrix}\lambda_1\mathbf{b} & & \mathbf{0}\\ & \lambda_2\mathbf{b} & \\ & & \ddots\\ \mathbf{0} & & \lambda_p\mathbf{b}\end{vmatrix}=\Pi\,|\,\lambda_i\mathbf{b}\,|\\&=\Pi\lambda_i^q\,|\,\mathbf{b}\,|=(\Pi\lambda_i)^q\,|\,\mathbf{b}\,|^p\\&=|\,\mathbf{a}\,|^q\,|\,\mathbf{b}\,|^p.\end{aligned}$$

IV. APPLICATIONS.

(i) *Direct powers.*

The r-fold continued direct product of a matrix with itself in the r-th direct power. We shall denote this by an exponent in brackets, thus

$$\underbrace{\mathbf{a}\times\mathbf{a}\times\ldots\times\mathbf{a}}_{r \text{ terms}}=\mathbf{a}^{[r]}.$$

If we define

$$\mathbf{a}^{[0]}=1, \quad \mathbf{a}^{[1]}=\mathbf{a},$$

then, for non-negative integers r, s, we have

$$\begin{aligned}\mathbf{a}^{[r]}\times\mathbf{a}^{[s]}&=\mathbf{a}^{[s]}\times\mathbf{a}^{[r]}\\ &=\mathbf{a}^{[r+s]}. \qquad \ldots\ldots\ldots\ldots (4.1)\end{aligned}$$

We now have the useful theorem that a direct power of a matrix product equals the matrix product of the direct powers.

$$(\mathbf{abc}\ldots)^{[r]}=\mathbf{a}^{[r]}\mathbf{b}^{[r]}\mathbf{c}^{[r]}\ldots . \qquad \ldots\ldots\ldots\ldots (4.2)$$

This is easily proved by induction, since the formula is certainly valid when $r=1$, and, if (4.2) holds,

$$\begin{aligned}(\mathbf{abc}\ldots)^{[r+1]}&=(\mathbf{abc}\ldots)\times(\mathbf{abc}\ldots)^{[r]}\\ &=(\mathbf{abc}\ldots)\times(\mathbf{a}^{[r]}\mathbf{b}^{[r]}\mathbf{c}^{[r]}\ldots)\\ &=(\mathbf{a}\times\mathbf{a}^{[r]})(\mathbf{b}\times\mathbf{b}^{[r]})(\mathbf{c}\times\mathbf{c}^{[r]})\ldots\\ &=\mathbf{a}^{[r+1]}\mathbf{b}^{[r+1]}\mathbf{c}^{[r+1]}\ldots .\end{aligned}$$

It may also be noted that

$$(\mathbf{a}')^{[r]}=(\mathbf{a}^{[r]})'.$$

(ii) *Potentiation of quadratic forms.*

Consider the quadratic form

$$\sum_{i,j} a_{ij}x_ix_j=\mathbf{x}'\mathbf{ax}.$$

Since this is a scalar, its direct powers are identical with its ordinary powers. Hence

$$\begin{aligned}(\mathbf{x}'\mathbf{ax})^r&=(\mathbf{x}'\mathbf{ax})^{[r]}\\ &=\mathbf{x}^{[r]\prime}\mathbf{a}^{[r]}\mathbf{x}^{[r]}. \qquad \ldots\ldots\ldots\ldots (4.3)\end{aligned}$$

Hence the r-th power of a quadratic form in the x_i, with matrix $\mathbf{a}$, is a quadratic form in the elements of $x^{[r]}$, with matrix $\mathbf{a}^{[r]}$.

(iii) *Vectorisation.*

It is sometimes necessary in matrix analysis to rearrange the elements of a matrix in the form of a vector. A convenient way of performing this operation on a matrix $\mathbf{a}$ is to assemble the columns of $\mathbf{a}'$, each beneath its predecessor, into a single column. This process we call " vectorisation ", and the resulting vector we denote by $\mathbf{a}^v$. The elements of $\mathbf{a}^v$ are the a_{ij} arranged in lexical order.

If, for example, $\mathbf{x}$ is a vector,

$$(\mathbf{xx}')^v=\mathbf{x}^{[2]}.$$

Another example is the expression for a quadratic form in $\mathbf{x}$ as a linear form in $\mathbf{x}^{[2]}$:

$$\mathbf{x}'\mathbf{ax}=\boldsymbol{\nu}'\mathbf{x}^{[2]}, \qquad \ldots\ldots\ldots\ldots (4.4)$$

where

$$\boldsymbol{\nu}=\mathbf{a}^v.$$

This allows us to express the r-th power of a quadratic form as a linear form in the elements of $x^{[2r]}$, since, using (4.4),

$$(\mathbf{x}'\mathbf{a}\mathbf{x})^r = \mathbf{v}^{[r]\prime}\mathbf{x}^{[2r]}.$$

The vectorised form of a matrix triple product is sometimes useful. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{h}$ and $\mathbf{k}$ be matrices such that

$$\mathbf{a} = \mathbf{h}\mathbf{b}\mathbf{k}'. \qquad (4.5)$$

Then

$$a_{ij} = \sum_{r,s} h_{ir} k_{js} b_{rs},$$

whence

$$\mathbf{a}^v = (\mathbf{h} \times \mathbf{k})\mathbf{b}^v. \qquad (4.6)$$

Hence, for example, the jacobian of the transformation (4.5) from $\mathbf{a}$ to $\mathbf{b}$ is

$$|\,\mathbf{h} \times \mathbf{k}\,| = |\,\mathbf{h}\,|^q\,|\,\mathbf{k}\,|^p,$$

where $\mathbf{h}$ is $(p \times p)$ and $\mathbf{k}$ is $(q \times q)$.

A special case of (4.5) is the matrix product:

$$\mathbf{a} = \mathbf{h}\mathbf{b}.$$

In this case

$$\mathbf{a}^v = (\mathbf{h} \times \mathbf{I})\mathbf{b}^v.$$

E. H. L.

1732. The plain fact is that St. Thomas [Aquinas] had not the intellectual equipment to deal with infinite series and we have this equipment today. They turn out to be much simpler than finite ones.—J. B. S. Haldane, *Everything has a history*. [Per Mr. H. V. Lowry.]

1733. Lambert was in correspondence with members of the Royal Society and employing part of his leisure [as a State prisoner] in formulating problems in equations the solution of which tried the brains of distinguished mathematicians. A letter of the Rev. Thomas Baker, addressed in September 4, 1678, from Bishop Nympton, where he was vicar, to John Collins, states: "Major-General Lambert, prisoner at Plymouth, hath sent me these problems to be solved. I desire the solution of them, having sent mine to him:

Prob. 1. $a : b :: c : d$

$aa + bb + cc + dd = 250$

$b + 5 = c$

$a + 9 = d$

Qu. a, b, c, d?

Prob. 2. $aa + bb + cc + dd = 756$

$b + 6 = c$

$b - 9 = a$

Qu. a, b, c, d?"

—W. H. Dawson, *Cromwell's Understudy* (1938), pp. 441–2.

1734. The despair of doing sums oppressed my mind so that all the previous labour spent on learning, whose most secret chamber I thought I knew already, seemed nothing, and to use Jerome's expression I who before thought myself a past master began again to be a pupil, until the difficulty solved itself, and at last, by God's grace, I grasped after incessant study the most difficult of all things, which they call fractions.—Aldhelm (later Bishop of Sherborne), letter to Haeddi, c. 680 A.D., quoted in Jarman, *Landmarks in the history of education*, p. 76. [Per Mr. J. W. Ashley Smith.]

FINITE GEOMETRY BY COORDINATE METHODS.

By T. J. Fletcher.

This paper was originally planned as an exposition of finite geometry in a form in which it could be used to illustrate all the work covered in the normal school geometry course. The beginning of it, however, was anticipated by an article for the *Gazette* by Mr. H. Martyn Cundy (1), which he has kindly allowed me to read before publication. This present paper shows how a coordinate system can be constructed for the geometry described in his article by using the field of residue classes (mod. 5); that is, the Galois field GF(5). A definition of distance is introduced which extends that adopted by Cundy to a form suited to the discussion of a wider range of theorems. This leads to the adoption of a complex number field, GF(5^2), which is first used to give a representation of the finite geometry analogous to the Argand diagram; and then to construct complex geometries. Finally, a brief indication is given of the methods of constructing other finite geometries with different numbers of points and lines.

Finite geometries were first described by O. Veblen and W. H. Bussey (2) in 1906; but readers of *Modern Mathematics for T. C. Mits*, by H. G. and L. R. Lieber (3) will have met the two-dimensional geometry with 25 points in the notation given in Table I.

A	*B*	*C*	*D*	*E*		*A*	*I*	*L*	*T*	*W*		*A*	*X*	*Q*	*O*	*H*
F	*G*	*H*	*I*	*J*		*S*	*V*	*E*	*H*	*K*		*R*	*K*	*I*	*B*	*Y*
K	*L*	*M*	*N*	*O*		*G*	*O*	*R*	*U*	*D*		*J*	*C*	*U*	*S*	*L*
P	*Q*	*R*	*S*	*T*		*Y*	*C*	*F*	*N*	*Q*		*V*	*T*	*M*	*F*	*D*
U	*V*	*W*	*X*	*Y*		*M*	*P*	*X*	*B*	*J*		*N*	*G*	*E*	*W*	*P*
		(i)						(ii)						(iii)		

Table I.

The twenty-five points lie in fives on thirty lines. There are six sets of five parallel lines (rows or columns of the same block in the table). Lines represented by a row and a column of the same block are perpendicular. The elementary geometrical properties of the system are discussed by Cundy.

Coordinates. A Cartesian coordinate system is introduced into Euclidean geometry by associating with each point an ordered pair of real numbers. The same is done here, but instead of using the field of real numbers the field of residue classes (mod 5) is used. The addition and multiplication tables of this field are given in Table II.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Table II.

Using block (i) of Table I, coordinates (x, y) are assigned to each point according to the scheme shown in Table III.

y					
4	A	B	C	D	E
3	F	G	H	I	J
2	K	L	M	N	O
1	P	Q	R	S	T
0	U	V	W	X	Y
	0	1	2	3	4 x

TABLE III.

Each line satisfies an equation of the first degree. Thus $QIUME$ (Table I, block iii) is $x \equiv y$ (mod 5); and $AILTW$ (Table I, block ii) is $x + 3y + 3 \equiv 0$ (mod 5).

The algebraic theory of equations of the first degree can be developed very easily. The relation between the coefficients of parallel lines is analogous to the relation in Cartesian geometry; but the relation between the coefficients of perpendicular lines will not be clear until later.

Equations of the second degree do not display such obvious analogies to the equations in Cartesian geometry. For example, it can be verified that the points on the circle $QTMXNW$ all satisfy

$$x^2 + 2y^2 + y + 1 \equiv 0 \pmod 5$$

and this is the equation of the circle. It follows from the distance formula adopted below that the equations of all circles may be written with the second degree terms $x^2 + 2y^2$. We must examine the number field we are using more closely in order to understand the appearance of these terms rather than the more familiar $x^2 + y^2$.

The metrical properties in Euclidean geometry are associated with the form $x^2 + y^2$, which is positive definite in the field of real numbers. The distance, s, between two points (x_1, y_1) and (x_2, y_2) is given by

$$s^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

But $x^2 + y^2$ is not "positive definite" in the field we are now using; it has factors $(x + 2y)(x + 3y)$. Properties based on this form cannot be expected to be analogous to metrical properties in Euclidean geometry. The form $x^2 + 2y^2$ has no factors in this field and is the form we need. Defining the distance, s, between two points (x_1, y_1) and (x_2, y_2) by the relation

$$s^2 \equiv (x_1 - x_2)^2 + 2(y_1 - y_2)^2 \pmod 5$$

we have a definition which is consistent with our previous conventions and which also enables " row-lengths " and " column-lengths " to be compared in a manner which was not previously possible. This formula distinguishes between the two, as " row-lengths " have squares which are quadratic residues, 1 or 4, and " column-lengths " have squares which are quadratic non-residues, 2 or 3. Thus, neglecting sense, " row lengths " are 1 or 2; and " column lengths " are $\sqrt{2}$ or $2\sqrt{2}$. The theorem of Pythagoras now holds in this geometry; the rectangles contained by intersecting chords of a circle are equal, when proper account is taken of sense; and Cundy's formulae for inversion in a circle can all be summarised in the more familiar form

$$r_1 r_2 \equiv a^2 \pmod 5,$$

where r_1 and r_2 are the distances of a pair of inverse points from the centre of the circle of inversion, whose radius is a.

The relation adopted for the definition of distance may seem somewhat arbitrary. A similar arbitrariness in the definition employed in Cartesian

geometry is overlooked because it is so familiar. From one point of view, metrical properties arise in a projective geometry by selecting an arbitrary conic as the metric conic and considering properties relative to it. When we select the particular (degenerate) conic whose line equation is

$$l^2 + m^2 = 0,$$

the Cartesian formula for distance, and all the associated results, follow. Metrical results of some kind are equally possible by selecting any other conic as the metric conic. In the finite geometry, we have selected as the metric conic the one whose line equation is $2l^2 + m^2 \equiv 0$. This is a degerate conic of the same type as the euclidean metric conic, and so the metrical properties resulting from it are analogous. The type of degeneracy is more important than any superficial numerical similarity to the form of the Cartesian equation.

A stage is reached in Cartesian geometry when, to get further understanding, it is useful to employ the complex variable, either in the Argand diagram, or by taking each of the coordinates (x, y) to be a complex number. The same stage has now been reached in our finite geometry ; and we shall use the Galois field of order 5^2 instead of the Galois field of order 5 which we have been using hitherto. The relevant general theorems on these fields are discussed by G. de B. Robinson (4).

In the real field there is no root to the equation $x^2 + 1 = 0$; and the complex field is the extension of the real field by adjoining i, where i is defined by the relation

$$i^2 + 1 = 0.$$

In the field GF(5) there are two roots to the equation $x^2 + 1 \equiv 0$, but there is no root to $x^2 + 2 \equiv 0$. We therefore extend the field by adjoining j, where j is defined by

$$j^2 + 2 \equiv 0 \pmod 5.$$

There is associated with each point (x, y) in the Argand diagram the complex number $x + iy$. With each point in this geometry is associated the complex number $x + jy$.

The modulus of z, $|z|$, is defined over the usual complex field by

$$|z|^2 = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 ;$$

and we define the modulus over GF(5^2) by

$$|z|^2 \equiv z\bar{z} \equiv (x + jy)(x + 4jy) \equiv x^2 + 2y^2 \pmod 5.$$

(Note, of course, that $4 \equiv -1 \pmod 5$). Hence the modulus of the affix of a point is equal to its distance from the origin. It is easy to verify that, with this definition, the modulus of a product is congruent to the product of the moduli of the factors. The inverse of z with respect to a circle centre at the origin and radius a is

$$z' \equiv a/\bar{z} \pmod 5.$$

As an example of the satisfying nature of this geometry we have the analogue of the theorem in the Argand diagram, that if four points are concyclic the cross-ratio of their affixes is real. By " real " in this field is meant, of course, a number of the form $x + j0$. The more usual proof of this, in the Argand diagram, depends on the equality of angles in the same segment and so will not apply here. But the method of inverting with respect to one of the four points, and expressing the condition for the inverses of the other three to lie on a real line (see Hardy (5), pp. 94, 100) applies equally well in this field of numbers.

4. *Projective geometry.* Many of the concepts of elementary geometry are discussed more easily in terms of the projective plane than in terms of the

euclidean plane. There is a 31-point domain related to the 25-point domain described above in the same way that the projective plane is related to the euclidean. To derive it a " *point at infinity* " may be postulated on each line, parallel lines possessing the same point at infinity, and then it may be shown that the points at infinity lie on a " *line at infinity* ". But it is more satisfactory to start afresh with the system shown in Table IV.

0	1	2	3	4	...	30
1	2	3	4	5	...	0
3	4	5	6	7	...	2
8	9	10	11	12	...	7
12	13	14	15	16	...	11
18	19	20	21	22	...	17

TABLE IV

Here is a geometry with thirty-one points, denoted by the numbers 0 to 30 ; and thirty-one lines, the columns of the table, with six points on each. The geometry is dual. The construction of the table is clear ; each row contains the numbers 0 to 30 in cyclic order. When one line is given the rest follow easily. The only problem arises in connection with the first line, and this is discussed later. A set of numbers such as (0, 1, 3, 8, 12, 18) which enables the incidence relations of the geometry to be described by this form of table we call a " *key* ".

Any one of the thirty-one lines of this geometry can be selected as a " *special line* ". On removing it we are left with twenty-five points, lying in fives on thirty lines. If we re-label them suitably, this is precisely the twenty-five point geometry we started with. This provides the pupil with an excellent illustration of what is meant by regarding the line at infinity from two points of view. From a projective point of view it is a line like any other ; it is also a special line because we have chosen it specially. The two finite geometries are readily accepted as equally abstract, and their mutual relations are easy to see. On the other hand, the beginner often has difficulty in seeing the relations between the Euclidean and projective planes in the same way. One has long been familiar and seems part of the natural world, while the other is new and seems much more artificial in consequence.

The 31-point system satisfies all the axioms required for the development of projective geometry, except the axiom of extension which is most usually taken. If we assume, therefore, the standard theorem of projective geometry, that any four points, no three of which are collinear, may be transformed by a collineation into any other four, no three of which are collinear, we see that the collineations which transform this geometry into itself form a group of order 372,000. For the first point may be transformed to any of 31 (including itself), the second to any of the remaining 30, the third to any of 25 (since it may not go to any of the six on the line joining the transforms of the first two) and the fourth to any of 16 (since it may not go to any of the points on the sides of the triangle formed by the transforms of the first three). Then

$$31 \times 30 \times 25 \times 16 = 372{,}000.$$

The number of collineations leaving a selected line invariant is

$$30 \times 25 \times 16 = 12{,}000,$$

and this is the order of the group of affine transformations. Cundy's groups of 1200 similarity and 300 congruence transformations are sub-groups of these.

The 31-point system is easily discussed using homogeneous coordinates. With each point is associated the number-triple (x, y, z) (not all zero), where (x, y, z) are elements of GF(5). The complex projective plane, which contains more points, can be discussed if (x, y, z) are taken to be complex numbers of the type described above.

It only remains to consider how many two-dimensional geometries of these types there are, and how they may be constructed. Simple methods of constructing Galois fields are described in Rouse Ball (6); and when a field is known it is clear that a corresponding geometry of number-pairs or number-triples can easily be constructed. The number of elements in a Galois field is necessarily of the form p^m, where p is prime and m is any integer. The geometries arising from the cases with $p=2$ are of less interest here because in them the diagonal points of a quadrangle are collinear, and so the analogies with Euclidean geometry are less close.

An alternative method is described by A. G. Walker (7). The " key " is constructed by finding a polynomial which satisfies certain conditions in the field in question, and using it to define a recurrence relation, which is used in turn to attach homogeneous coordinates to the points in the geometry. The geometry can then be displayed after the manner of Table IV. Keys for the first few cases are listed in Table V. Walker's paper also describes how finite geometries may be constructed in any number of dimensions by similar methods.

TABLE V

$n=p^m$	N	*Key*
3	12	0, 1, 3, 9
5	30	0, 1, 3, 8, 12, 18 or 0, 1, 3, 10, 14, 26
7	56	0, 1, 3, 13, 32, 36, 43, 52
3^2	90	0, 1, 3, 9, 27, 49, 56, 61, 77, 81
11	132	0, 1, 3, 15, 46, 71, 75, 84, 94, 101, 112, 128

$N=n^2+n$; $n+1=$number of points on each line ;
$N+1=$number of points in all.

BIBLIOGRAPHY.

1. H. Martyn Cundy, " 25-point Geometry," *Math. Gazette*, XXXVI (1952), p. 158.
2. O. Veblen and W. H. Bussey, *Trans. American Math. Soc.*, VII (1906) p. 246.
3. H. G. and L. R. Lieber, *Modern Mathematics for T. C. Mits.* (London, 1946).
4. G. de B. Robinson, *The Foundations of Geometry* (Toronto, 1940).
5. G. H. Hardy, *Pure Mathematics*, 7th edition (Cambridge, 1938).
6. W. W. Rouse Ball, *Mathematical Recreations*, 11th edition (London, 1939).
7. A. G. Walker, *Edinburgh Mathematical Notes* (May, 1947).

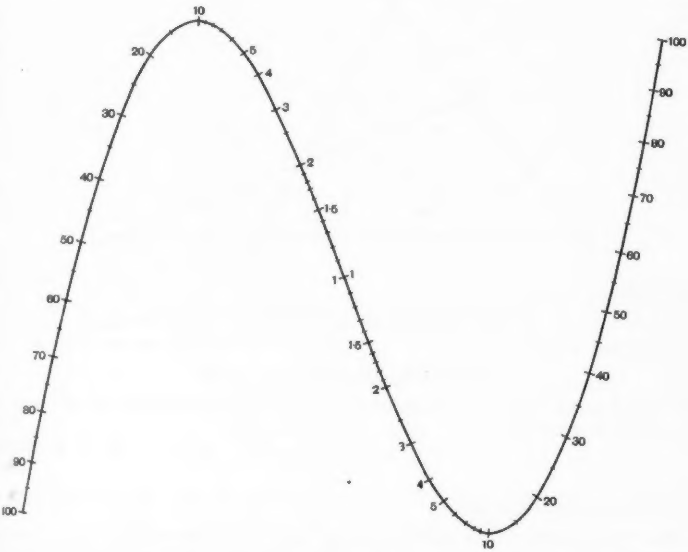
T. J. F.

1735. A Tass Agency message broadcast by Moscow radio states that, contrary to the opinion of western scientists, the equator is not a circle but an ellipsoid. Studies by Professor Alexander Izotov have established that the calculations of the radius of the equator by the American scientist Heyford and the German scientist Friedrich Bessel are wrong, the message said.—*The Times*, 16th February, 1952. [Per Dr. T. J. Willmore and Mr. P. S. W. MacIlwaine.]

MATHEMATICAL NOTES

2315. *A single-scale nomogram.*

Since the equation $x^3+ax+b=0$ has zero for the sum of its roots, the x-coordinates of the three intersections of the line $y=mx+c$ and the curve $y=x^3+px+q$ add to zero. Thus a curve $y=x^3+px+q$, graduated according



to its x-coordinates, provides a single-scale nomogram for simple addition and subtraction. Graduated according to the antilogarithms of its x-coordinates, it becomes a nomogram for multiplication and division. In fact, appropriate graduation will give a nomogram for performing any operation which may be executed by a suitably graduated slide-rule.

Practical problems in the construction of such a nomogram are (a) selection of a suitable cubic curve, *e.g.* in order to give a good " cut " in the majority of cases, (b) selection of suitable scales for the x- and y-axes, (c) selection of tbe range of x-values to be used, (d) selection of the range of graduations to he displayed on the curve, *e.g.* the base of the antilogarithms in the example given below.

The diagram is for multiplication and division. For the former, join two numbers on the same half of the curve by a straight line, and read its third intersection with the (other half of the) curve. For division, connect numbers on opposite halves of the curve.

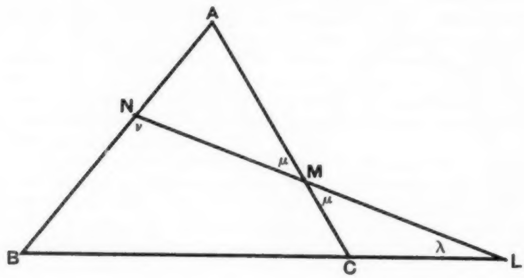
Those who listened to Professor Brodetsky's lecture on " nomography " at Birmingham in 1949 will notice that the a-, b- and x- curves are of one and the same, and may be interested in investigating the possibility connecting more than three variables by means of a family of cubic curves.

RICHARD K. GUY.

2316. *Menelaus' Theorem.*

With the notation of the figure, by the sine formula applied to the triangle ANM,

$$MA/AN = \sin \nu/\sin \mu.$$



Similarly,

$$NB/BL = \sin \lambda/\sin \nu, \quad LC/CM = \sin \mu/\sin \lambda.$$

Multiplying, we get

$$MA \,.\, NB \,.\, LC = AN \,.\, BL \,.\, CM.$$

The usual minus sign is, of course, to be supplied, since either one or three of the ratios is negative.

R. C. Everitt.

2317. *Bending the beam.*

The " engineering " method of solving beam problems is often received by students with more scepticism than it deserves. The use of the Euler-Bernoulli approximation $M = EI\, d^2y/dx^2$ certainly needs care, but even beyond this point there are at least two further questions that are perhaps more often sensed than expressed: Is the concept of bending moment a sound one? How are the equations of the method related to the more general stress equilibrium relations $\partial T_{\alpha i}/\partial x_\alpha = -W_i$ and $T_{ij} = T_{ji}$? (We write T_{ij} for the stress tensor and W_i for the body force per unit volume.)

The Engineering Equations.

Consider a plane cross-section of a beam of uniform cross-section. We replace all the stress forces acting *on* the left-hand side *from* the right by a force $\mathbf{F}$ through the centroid of the cross section and a couple $\mathbf{M}$ about this point. We define these quantities to be the *section force* and the *section couple.* When we consider the equilibrium of a small element of the beam we must take into account not only these section quantities but also the resultant of the body forces and the external forces (including loads, etc.) acting on the lateral surface of the element. Let this resultant be equivalent to a force $\mathbf{A}\,dx$ acting through the centroid of the element and a couple $\mathbf{m}\,dx$ about this point.

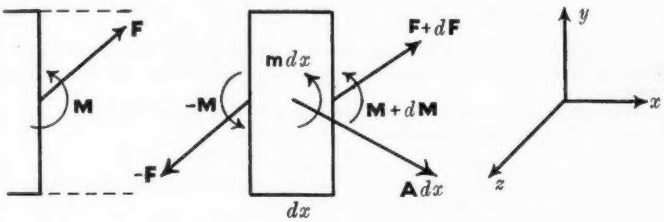
Assume that all forces involved are continuous functions of position. Then we easily obtain that, for equilibrium,

$$d\mathbf{F}/dx = -\mathbf{A}, \quad\text{...(1)}$$

$$d\mathbf{M}/dx = -\mathbf{i}\wedge\mathbf{F} - \mathbf{m}, \quad\text{.................................(2)}$$

and combining these two equations,

$$d^2\mathbf{M}/dx^2 = \mathbf{i}\wedge\mathbf{A} - d\mathbf{m}/dx. \quad\text{..............................(3)}$$



Connection with the General Theory.

The standard method for getting the stress equilibrium relations is to write down the condition that matter within any closed surface whose direction cosines are l_i be in equilibrium. Thus,

$$\int l_\alpha T_{\alpha i}\, dS + \int W_i\, dV = 0, \quad\text{..............................(4)}$$

and

$$\int E_{irs} x_r l_\alpha T_{\alpha s}\, dS + \int E_{irs} x_r W_s\, dV = 0. \quad\text{.......................(5)}$$

Now, equations (4) and (5) are precisely equivalent to equations (1) and (2) respectively : they differ only in notation. By applying Green's theorem, we get $\partial T_{\alpha i}/\partial x_\alpha = -W_i$ from (4), and $T_{ij} = T_{ji}$ from (4) and (5) together. We therefore conclude that, properly written, the beam bending formulae are exact consequences of the stress equilibrium relations and possess the same generality. Indeed, it is clear that analagous formulae would hold for any family of surfaces cutting an elastically stressed body. Nor is there any need to choose the line of centroids to define **M**, **F**, etc.

Usual Formulation.

If we take the y component of equation (1) and the z components of (2) and (3), we get the standard equations *except for additional terms involving* **m**. However, it is easy to see that $\mathbf{m} = 0$ for the type of problem usually considered. An example for which this is not true is provided by a beam clamped at its lower end, loaded on its top surface and projecting at an angle to the horizontal.

R. O. DAVIES

2318. *Application of Vector Methods in Three-dimensional Statics.*

In learning vectorial methods in Mechanics, one of the chief difficulties encountered by the student is in the application of an unfamiliar notation to definite problems. In order to overcome this difficulty I have found that the following technique helps to bridge the gap without introducing any ideas which later must be discarded.

The fundamental conditions of equilibrium are

$$\Sigma \mathbf{P} = 0 \quad \text{and} \quad \Sigma \mathbf{r} \wedge \mathbf{P} = 0.$$

Hence, by choosing a positive triad of unit vectors as in Milne's *Vectorial Mechanics*, we can take these as coordinate axes and thus make up an array for each force as shown

$$\begin{Bmatrix} \mathbf{r} \\ \mathbf{P} \\ \mathbf{r} \wedge \mathbf{P} \end{Bmatrix} \equiv \begin{Bmatrix} x, & y, & z \\ X, & Y, & Z \\ yZ - zY, & zX - xZ, & xY - yX \end{Bmatrix}.$$

The general equation of equilibrium or equivalence for the system of forces can then be written down and the relevant parts picked out for the solution of the problem.

I give below the solution of two problems which have been set in London University Degree Examinations.

Ex. 1. Find two forces, one in the plane $lx + my + nz = 0$ and the other passing through the point $(0, a, 0)$, which will be equivalent to a wrench (R, pR) acting along the z-axis.

The moment of the first force about the origin is $M\mathbf{n}$, where $\mathbf{n}$ is a unit vector along the normal to the plane which contains it. Hence the equation of equivalence is

$$\begin{Bmatrix} \mathbf{r} \\ \mathbf{P} \\ \mathbf{r} \wedge \mathbf{P} \end{Bmatrix} \equiv \begin{Bmatrix} x, & y, & z \\ X, & Y, & Z \\ lM, & mM, & nM \end{Bmatrix} + \begin{Bmatrix} 0, & a, & 0 \\ X_1, & Y_1, & Z_1 \\ aZ_1, & 0, & -aX_1 \end{Bmatrix} = \begin{Bmatrix} . & . & . \\ 0, & 0, & R \\ 0, & 0, & pR \end{Bmatrix}.$$

From the y, x and z components of $\mathbf{r} \wedge \mathbf{P}$ respectively, we have

$$mM = 0, \quad \text{and so} \quad M = 0 \quad (\text{since } m \neq 0),$$
$$aZ_1 = 0, \quad \text{and so} \quad Z_1 = 0 \quad (\text{since } a \neq 0),$$
$$-aX_1 = pR, \quad \text{and so} \quad X_1 = -pR/a.$$

From the components of $\mathbf{P}$ we now have the following:

$$X = pR/a, \quad Y = -Y_1, \quad Z = R.$$

Since the first force is in the plane $lx + my + nz = 0$,

$$lpR/a - mY_1 + nR = 0,$$

or

$$Y_1 = (lp + na)R/ma.$$

Because $M = 0$, the first force acts through the origin. Hence it is

$$\{pR/a, \quad -(lp + na)R/ma, \quad R\}$$

acting in the line

$$\frac{x}{p/a} = \frac{y}{-(lp + na)/ma} = \frac{z}{1}.$$

The second force is

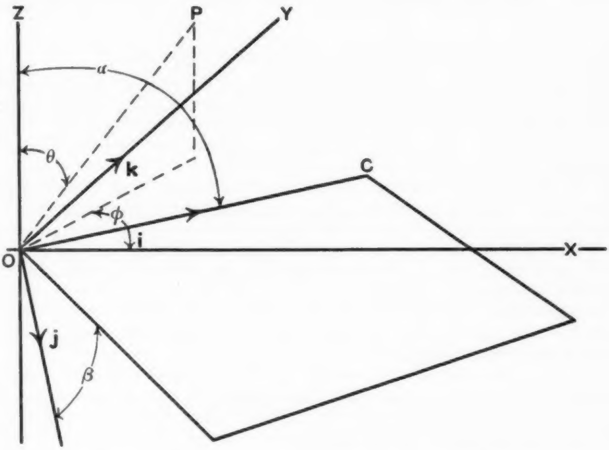
$$\{-pR/a, \quad (lp + na)R/ma, \quad 0\}$$

acting in the line

$$\frac{x}{-mp} = \frac{y - a}{lp + na} = \frac{z}{0}.$$

Ex. 2. The position of a thin uniform rectangular trapdoor of weight W, free to turn about the fixed edge OC, is given by angles α, β, where α is the angle between OC and the upward vertical OZ and β is the angle between the plane of the trapdoor and the plane COZ. It is kept in this position by wind pressure equivalent to a normal force at the

centre of gravity of amount W/k times the cosine of the angle between the normal and the wind direction OP. If OP makes an angle θ with OZ in a plane ZOP making an angle ϕ with the plane ZOC, prove that $\cot\beta = \cos\alpha\cot\phi - \sin\alpha\operatorname{cosec}\phi\,(\cot\theta - k\operatorname{cosec}\theta)$.



Let ZOX be the vertical plane containing OC and let OY be into the plane of the paper.

Take unit vectors $\mathbf{i}$ along OC, $\mathbf{j}$ perpendicular to OC in plane ZOC and $\mathbf{k}$ along OY.

Since we are only concerned about the moments of the forces about the hinge, *i.e.* $\mathbf{i}$, we can take a wind force W/k parallel to OP to satisfy the conditions of the problem. Its components along OX, OY, OZ are therefore $W/k\sin\theta\cos\phi$, $W/k\sin\theta\sin\phi$, $W/k\cos\theta$. Furthermore, in $\mathbf{r}\wedge\mathbf{P}$ we need not work out the $\mathbf{j}$ and $\mathbf{k}$ components. We therefore get the following equation of equilibrium :

$$\begin{Bmatrix} \mathbf{r} \\ \mathbf{P} \\ \mathbf{r}\wedge\mathbf{P} \end{Bmatrix} \equiv \overset{\textit{Hinge}}{\begin{Bmatrix} x, & 0, & 0 \\ X, & Y, & Z \\ 0, & M, & N \end{Bmatrix}} + \overset{\textit{Weight}}{\begin{Bmatrix} a, & b\cos\beta, & b\sin\beta \\ -W\cos\alpha, & W\sin\alpha, & 0 \\ -Wb\sin\alpha\sin\beta, & M_1, & N_1 \end{Bmatrix}}$$

$$+\overset{\textit{Wind}}{\begin{Bmatrix} a, & b\cos\beta, & b\sin\beta \\ \begin{array}{l}(\sin\theta\cos\phi\sin\alpha \\ \quad +\cos\theta\cos\alpha)W/k,\end{array} & \begin{array}{l}(\sin\theta\cos\phi\cos\alpha \\ \quad -\cos\theta\sin\alpha)W/k,\end{array} & \sin\theta\sin\phi\,.\,W/k \\ \begin{array}{l}(\cos\beta\sin\theta\sin\phi \\ \quad -\sin\beta\sin\theta\cos\phi\cos\alpha \\ \quad +\sin\beta\cos\theta\sin\alpha)Wb/k,\end{array} & M_2, & N_2 \end{Bmatrix}} = 0.$$

From the first components of $\mathbf{r}\wedge\mathbf{P}$ we therefore get

$$Wb \sin\alpha \sin\beta = Wb(\cos\beta \sin\theta \sin\phi - \sin\beta \sin\theta \cos\phi \cos\alpha + \sin\beta \cos\theta \sin\alpha)/k,$$

whence $$k \sin\alpha \operatorname{cosec}\phi \operatorname{cosec}\theta = \cot\beta - \cot\phi \cos\alpha + \sin\alpha \operatorname{cosec}\phi \cot\theta$$

or $$\cot\beta = \cot\phi \cos\alpha - \sin\alpha \operatorname{cosec}\phi(\cot\theta - k \operatorname{cosec}\theta).$$

R. H. PEACOCK.

2319. *Note on a class of problems in elementary dynamics.*

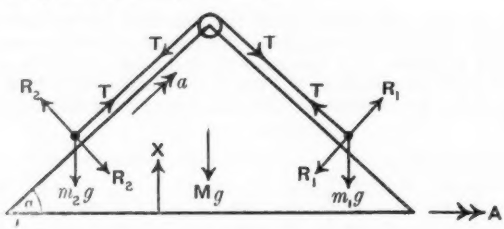
Recent experience in teaching dynamics to first-year undergraduates suggests that a certain class of problems is being taught rather badly in schools. I refer to problems involving two or more particles when one is not required to find the values of the internal forces of the system, but merely asked to find, for example, the acceleration of one of the particles. There is a fairly widespread tendency to obtain the equation of motion of each separate particle of the system, thereby introducing a large number of unknowns which must be subsequently eliminated. But the required solution may often be obtained without the introduction of irrelevancies by considering the motion of the centre of gravity of the system, and by making use of the principle of conservation of energy.

As a representative example of the class of problems under discussion, consider the worked example on page 198 of *Mechanics*, by Palmer and Snell, University of London Press.

"*A smooth isosceles wedge of mass M is placed on a smooth table, and carries a small smooth pulley at its summit. Two masses, m_1 and m_2, are attached to the ends of a string which passes over the pulley. Show that the acceleration with which the string passes over the pulley is*

$$\frac{m_1 - m_2}{m_1 + m_2} \cdot \frac{M + m_1 + m_2}{M + (m_1 + m_2)\sin^2\alpha} \cdot g \sin\alpha, \quad \text{.....................(1)}$$

where α is the angle at the base of the wedge."



The solution given by Palmer and Snell is outlined below. Relative to the wedge, let m_1 and m_2 have acceleration a down and up the wedge respectively: and let A be the acceleration of the wedge relative to the table. Let T be the tension in the string, and R_1, R_2 the reaction between M and m_1, m_2 respectively. Let X be the reaction between the wedge and the table.

By considering the motion of m_1, m_2 and M separately, the following six equations are obtained :

$$T - m_2 g \sin\alpha = m_2(a + A\cos\alpha),$$
$$m_2 g \cos\alpha - R_2 = m_2 A \sin\alpha,$$
$$m_1 g \sin\alpha - T = m_1(a + A\cos\alpha),$$
$$R_1 - m_1 g \cos\alpha = m_1 A \sin\alpha,$$

$$R_2 \sin \alpha - R_1 \sin \alpha = MA,$$
$$2T \sin \alpha + (R_1 + R_2) \cos \alpha + Mg - X = 0.$$

Actually, only the first five equations are used ; and by eliminating A, T, R_1 and R_2 from these equations, the required solution (1) is obtained.

The alternative solution below is not only simpler, but makes use of important dynamical principles.

Let y be the displacement of m_1 relative to the wedge, and let x be the displacement of the wedge relative to the table. Since the external forces have no horizontal component, it follows that if the system be initially at rest, then in the subsequent motion the centre of gravity of the system has no horizontal motion. This gives

$$Mx + (m_1 + m_2)(x + y \cos \alpha) = 0$$

during the whole motion, *i.e.*

$$x = -(m_1 + m_2) y \cos \alpha / (M + m_1 + m_2). \qquad \text{......................(2)}$$

The same relation is satisfied by the velocities $\dot{x}$, $\dot{y}$ and the accelerations, $\ddot{x}$ and $\ddot{y}$; this follows immediately by differentiating (2).

The principle of energy gives a further relation, *viz.* rate of increase of kinetic energy is equal to rate of loss of potential energy. This gives

$$\frac{d}{dt}\left[\tfrac{1}{2} M \dot{x}^2 + \tfrac{1}{2}(m_1 + m_2)(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y} \cos \alpha)\right] = \frac{d}{dt}\left[(m_1 - m_2) g y \sin \alpha\right],$$

i.e.

$$(M + m_1 + m_2)\dot{x}\ddot{x} + (m_1 + m_2)[\dot{y}\ddot{y} + (\ddot{x}\dot{y} + \dot{x}\ddot{y}) \cos \alpha] = (m_1 - m_2) g \dot{y} \sin \alpha. \quad \text{...(3)}$$

If we use in equation (3) the relations between $\dot{x}$, $\dot{y}$ and between $\ddot{x}$, $\ddot{y}$ given by (2), we readily obtain $\ddot{y}$ in the form given by (1).

Our method depends effectively upon the solution of two equations in two unknowns rather than on five equations in five unknowns ; and since the internal forces T, R_1, R_2 have not been introduced, the question of their elimination does not arise.

In fairness to Palmer and Snell, it should be mentioned that the section of their book dealing with the motion of the centre of gravity of two or more particles is dealt with immediately after their worked example. Nevertheless I think it unfortunate that no indication is given that this *worked example* is best treated by other methods.

T. J. WILLMORE.

2320. *The value of a product whose factors are liable to error.*

The length and breadth of a rectangle, measured to the nearest millimetre with a metre rule, are stated to be 3·6 and 2·3 cm. We wish to compute its area. We find that $3{\cdot}6 \times 2{\cdot}3 = 8{\cdot}28$. But we must not give the answer to more than two significant figures ; so we give it as 8·3 sq. cm. If the measurements had instead been 3·5 and 2·5, we should have found the product 8·75 ; ought we to call this 8·7 or 8·8? Common practice, at any rate in the schools, seems to be to regard the choice as arbitrary, and to select (as a matter of convention) 8·8. The theorem given below suggests that the choice is not arbitrary, and that the better answer is in fact 8·7.

Let a_0, liable to an error not more than α in either direction, and b_0, liable to an error not more than β, be values assigned by measurement.

Let a and b be the corresponding true values ; so that *all* we know about a and b is that

$$\begin{cases} a_0 - \alpha \leqslant a \leqslant a_0 + \alpha, \\ b_0 - \beta \leqslant b \leqslant b_0 + \beta. \end{cases}$$

Then ab, the true value of the product, is more likely to be less than a_0b_0 than to be greater.

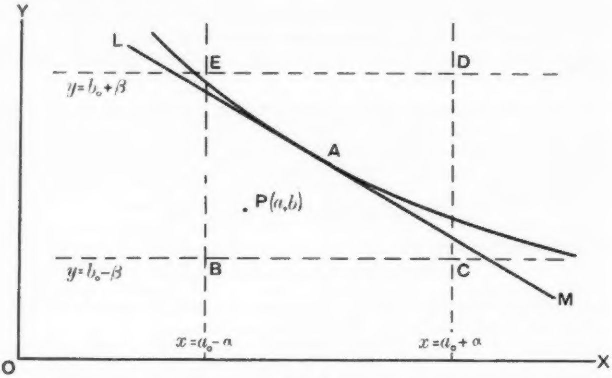


FIG. 1.

The curve is $xy = a_0b_0$ and A is the point (a_0, b_0).

The proof is very simple.

(i) Let OX, OY be two perpendicular axes. Then every possible pair of values for a and b can be uniquely represented by a point $P(a, b)$ lying within the rectangle $BCDE$, formed by the lines $x = a_0 \pm \alpha$, $y = b_0 \pm \beta$. The centre of this rectangle is the point $A(a_0, b_0)$. (See Fig. 1.)

(ii) $ab \gtreqless a_0b_0$ according as P lies above, on or below the rectangular hyperbola $xy = a_0b_0$. For, if P is above the hyperbola (Fig. 2), then

$$ab = \text{rectangle } OQPR > \text{rectangle } OQ'P'R = a_0b_0.$$

And if P is below the hyperbola (Fig. 3), then

$$ab = \text{rectangle } OQPR < \text{rectangle } OQ'P'R = a_0b_0.$$

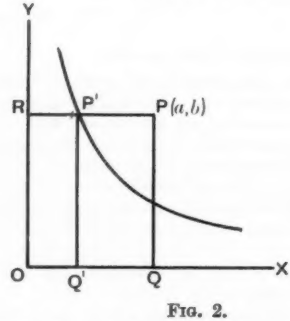


FIG. 2.

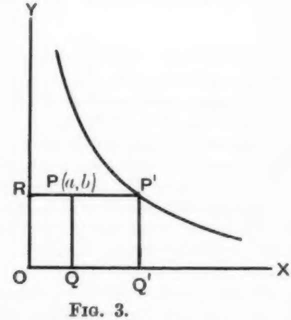


FIG. 3.

(iii) The hyperbola divides the rectangle $BCDE$ (Fig. 1) into two parts and the chance that $ab < a_0b_0$ is the chance that P shall lie in the lower of these two parts.

(iv) All that remains to be proved is that the lower part is the greater. Now the hyperbola is entirely above its tangent, LM, at A; and LM, passing through the centre of the rectangle, bisects it.

This completes the proof. L. W. H. HULL.

2321. *Freak goals as an illustration of dynamical principles.*

Arising out of controversies caused when a football hits the crossbar, bounces behind the goal line and then rebounds into play, an interesting problem in dynamics seems worth investigating.

To simplify the problem, let us consider the case of the ball travelling horizontally just before impact with the underside of the crossbar and rebounding vertically downwards. Let the coefficients of friction at the bar and the ground be μ, μ', and the coefficients of restitution e, e' respectively. At the moment of striking let the horizontal velocity be u and the angle between the radius of contact and the horizontal be θ.

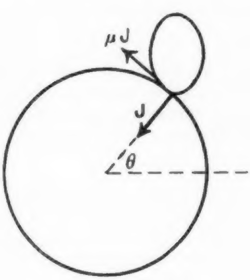


FIG. 1.

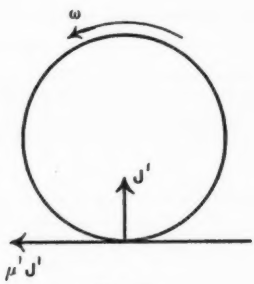


FIG. 2.

If we now take the case when the ball slips at the point of contact the impulses will be J and μJ as shown (Fig. 1). Let the velocity after impact be v vertically. Hence

$$J(\cos\theta + \mu\sin\theta) = mu, \quad \text{.............................(1)}$$

$$J(\sin\theta - \mu\cos\theta) = mv. \quad \text{.............................(2)}$$

The normal relative velocity equation gives

$$eu\cos\theta = v\sin\theta. \quad \text{...................................(3)}$$

Divide (2) by (1),

$$\frac{v}{u} = \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}.$$

Substituting for $\cos\theta$ from (3), we find, on removing the factor $\sin\theta$,

$$\frac{v}{u} = \left(1 - \frac{\mu}{e}\frac{v}{u}\right) \Big/ \left(\frac{1}{e}\frac{v}{u} + \mu\right).$$

Hence

$$\frac{1}{e}\left(\frac{v}{u}\right)^2 + \mu\left(1 + \frac{1}{e}\right)\frac{v}{u} - 1 = 0.$$

The positive value of v is therefore given by

$$v = \tfrac{1}{2}u[-\mu(1+e) + \sqrt{\{\mu^2(1+e)^2 + 4e\}}]. \quad \text{.....................(4)}$$

The ball now falls vertically a distance h and strikes the ground with a velocity

v_1 given by

$$v_1^2 = v^2 + 2gh. \quad\quad (5)$$

In addition to the above linear motion the ball has also acquired a spin ω in an anti-clockwise direction because of the impulse μJ. If this spin is sufficient to cause slipping at the point of contact with the ground then the impulses will be J' and $\mu' J'$ as shown (Fig. 2). Let the vertical and horizontal components of the velocity of rebound be v_2, u_2. Then we have

$$J' = m(v_1 + v_2), \quad\quad (6)$$

$$\mu' J' = m u_2, \quad\quad (7)$$

$$e' v_1 = v_2. \quad\quad (8)$$

Dividing (6) by (7),

$$1/\mu' = (v_1 + v_2)/u_2. \quad\quad (9)$$

Substituting for v_1 from (8),

$$1/\mu' = v_2(1 + e')/u_2 e'.$$

The ball therefore rebounds away from the goal at an angle

$$\tan^{-1}\{e'/\mu'(1 + e')\}$$

to the horizontal. This shows that the angle at which the ball bounces is independent of the initial conditions provided that there is sufficient impulsive friction at the crossbar to provide an angular velocity which will be great enough to cause slipping on contact with the ground.

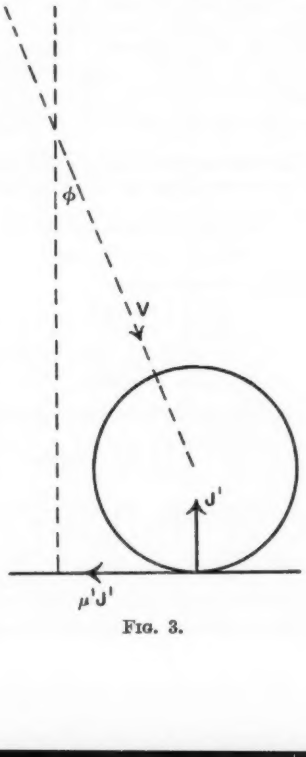


Fig. 3.

If we are given the initial velocity, then equations (4), (5), (8) and (9) will enable us to calculate the velocity after bouncing on the ground.

If slipping does not occur during impact then the problem becomes indeterminate. The frictional impulses will be reduced and the angle between the direction of rebound and the horizontal will be increased.

We can now analyse the original problem in the light of these results. If we are only concerned with the angle of rebound then the conditions at the ground are the only ones that matter (Fig. 3).

Let the velocity of impact be V at an angle ϕ to the vertical. The equations governing the impact, using the same notation as before, are as follows :

$$J' = m(V\cos\phi + v_2), \qquad (10)$$

$$\mu' J' = m(V\sin\phi + u_2), \qquad (11)$$

$$e' V\cos\phi = v_2. \qquad (12)$$

Dividing (10) by (11) and substituting for V from (12) :

$$\frac{1}{\mu'} = \frac{v_2(e'+1)}{v_2\tan\phi + e'u_2}.$$

Hence

$$\frac{v_2}{u_2} = \frac{e'}{\mu'(1+e') - \tan\phi} = \tan\alpha,$$

where α is the angle which the direction of rebound makes with the horizontal. It will therefore rebound out of goal provided that

$$\mu'(1+e') > \tan\phi.$$

As the height of the goal is 8 feet, the value of $\tan\phi$ will be approximately $\frac{1}{8}$ and so this last inequality is quite easily satisfied.

Finally, it is sometimes stated that it is impossible to have frictional impulses but the statement needs qualifying. If the impact is such that it does not affect the normal reaction then frictional forces can be neglected during the time of the impact because it is so short. If, however, the impact increase the normal reaction then a frictional impulse will be called into play which will have a proportionate effect during the time of the impact.

R. H. PEACOCK.

2322. *The half-angle formula.*

The formula

$$\tan\tfrac{1}{2}A = \sqrt{\{(s-b)(s-c)/s(s-a)\}}$$

is generally obtained from the cosine rule by first establishing the formulae for $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ and forming the appropriate ratio. The following proof, which may be found in some text-books, though I have not seen it, has a conciseness which should commend itself to certain examinees in mathematics.

$$\cos A = (b^2 + c^2 - a^2)/2bc,$$

so that, writing $t = \tan\frac{1}{2}A$,

$$(1-t^2)/(1+t^2) = (b^2+c^2-a^2)/2bc$$

and hence

$$t^2\{(b+c)^2 - a^2\} = a^2 - (b-c)^2.$$

Thus

$$t^2 \,.\, 2s \,.\, 2(s-a) = 2(s-b) \,.\, 2(s-c),$$

whence

$$\tan\tfrac{1}{2}A = t = \sqrt{\{(s-b)(s-c)/s(s-a)\}},$$

since t is necessarily positive as $\frac{1}{2}A < 90°$.

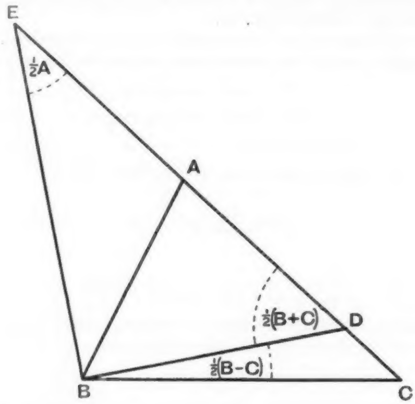
W. CRAIG.

2323. *The formula for tan* $\frac{1}{2}(B-C)$.

Let $b>c$. With centre A and radius AB cut CA internally and externally at D and E respectively. Join BD, BE. Then

$$DC=b-c, \quad EC=b+c,$$

$$\angle E=\tfrac{1}{2}A, \quad \angle EBD=90^\circ, \quad \angle DBC=\tfrac{1}{2}(B-C), \quad \angle ADB=\tfrac{1}{2}(B+C).$$



From the triangle DBC,

$$BC/(b-c)=\cos\tfrac{1}{2}A/\sin\tfrac{1}{2}(B-C)\,;$$

from the triangle EBC,

$$BC/(b+c)=\sin\tfrac{1}{2}A/\cos\tfrac{1}{2}(B-C).$$

Hence

$$\frac{b-c}{b+c}\cot\tfrac{1}{2}A\,.\,\cot\tfrac{1}{2}(B-C)=1,$$

or

$$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\,.\,\cot\tfrac{1}{2}A.$$

Aliter.

$$\begin{aligned}\Delta BDC/\Delta BCE &= (b-c)/(b+c)\\ &= \tfrac{1}{2}a\,.\,BD\,.\,\sin\tfrac{1}{2}(B-C)/\tfrac{1}{2}a\,.\,BE\,.\,\sin\{90^\circ+\tfrac{1}{2}(B-C)\}\\ &= (BD/BE)\,.\,\{\sin\tfrac{1}{2}(B-C)/\cos\tfrac{1}{2}(B-C)\}\\ &= \tan\tfrac{1}{2}A\tan\tfrac{1}{2}(B-C).\end{aligned}$$

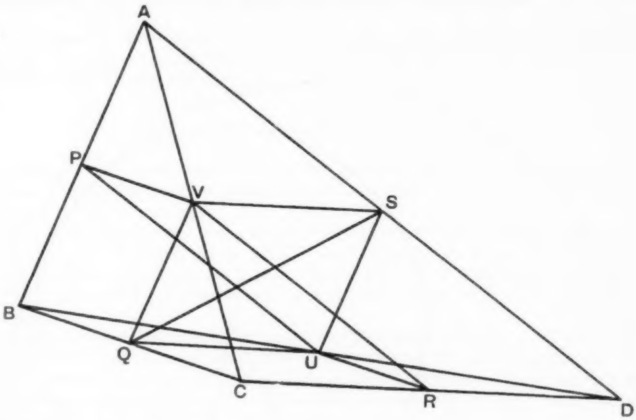
Thus $$\tan\tfrac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\tfrac{1}{2}A.$$

W. CRAIG.

2324 *Construction of a quadrangle.*

In Note 2156 (Vol. XXXIV, No. 309). Mr. Curnow has shown a construction for a quadrangle, given the sides and the distance of the midpoints of two opposite sides. I recall that M. Thébault proposed a very similar problem in the *American Mathematical Monthly* (E 707), and I gave the following solution

Let $ABCD$ be the quadrangle, $AB=a$, $BC=b$, $CD=c$, $DA=d$, $BD=u$, $AC=v$; P, Q, R, S, U, V the midpoints of a, b, c, d, u, v; $PR=m$, $QS=l$. The given data are a, b, c, d, l. We first construct the parallelogram $SUQV$, of which we know $US=QV=\frac{1}{2}a$, $QU=SV=\frac{1}{2}c$, and the diagonal $QS=l$. Then we determine the points P, R, knowing that $PU=RV=\frac{1}{2}d$, $PV=RU=\frac{1}{2}b$. Through P and S we draw the parallels to US and VR which fix the vertex A. On AS we determine D from the fact that $SD=AS$, on AP the vertex B, since $AP=PB$, and on AV the vertex C, since $AV=VC$.



This procedure enables us to solve more difficult problems, for example, to construct a quadrangle given a, c, u, l and the angle between u and v.

JOSEPH LANG, Prague.

2325 *Some oblique angles are right angles.*

Three circles whose centres are A, B, C and radii a, b, c touch each other externally, and also touch internally a circle centre R and radius r.

The figure is accurately constructed for the case $a=7$, $b=10$, $c=18$, and $\angle CAB$ is an acute angle. The figure could equally well have been drawn with $\angle CAB$ an obtuse angle. We shall " prove " that $\angle CAB$ is a right angle.

Then, using Hero's formula for the area of a triangle,

$$\triangle RBC = \sqrt{\{rbc(r-b-c)\}},$$
$$\triangle RCA = \sqrt{\{rca(r-c-a)\}},$$
$$\triangle RAB = \sqrt{\{rab(r-a-b)\}},$$
$$\triangle ABC = \sqrt{\{(a+b+c)abc\}}.$$

But we have

$$\triangle ABC + \triangle RBC = \triangle RCA + \triangle RAB.$$

Hence,

$$\sqrt{\{(a+b+c)abc\}} + \sqrt{\{rbc(r-b-c)\}} = \sqrt{\{rca(r-c-a)\}} + \sqrt{\{rab(r-a-b)\}}.$$

This equation, when rationalized, is of the eighth degree in r, but not all the roots of the rationalized equation are roots of the equation given.

Fortunately, a root of the equation is obvious.

This root is $r=a+b+c$.

Hence

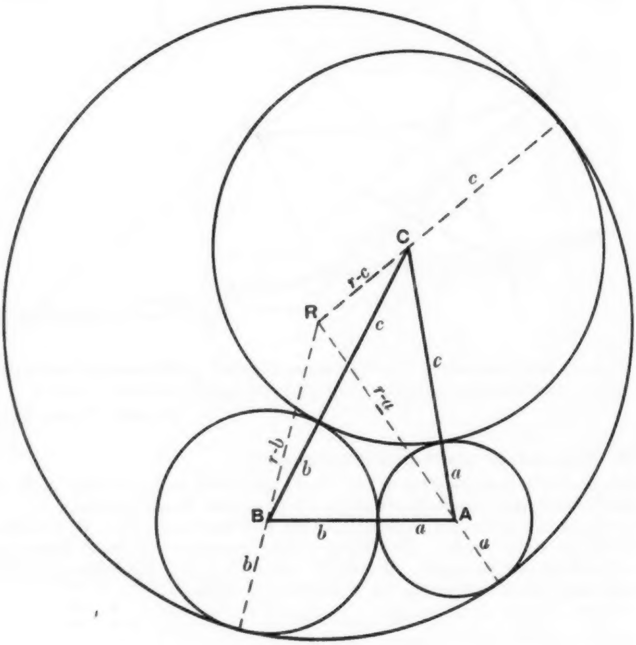
$$RB=r-b=a+c=AC,$$
$$RC=r-c=a+b=AB.$$

Therefore, the quadrilateral $ABRC$ has its opposite sides equal, and is a parallelogram.

But $$RA=r-a=b+c=BC.$$

Hence the parallelogram $ABRC$ has equal diagonals, and is therefore a rectangle.

Therefore $\angle CAB$ is a right angle.



Unlike most of the geometrical " proofs " of this kind, the fallacy does *not* lie in an inaccurate diagram.

It is perfectly true that $r=a+b+c$ is a solution of the equation, and the equation itself is perfectly correct for the case illustrated. The fallacy lies in the fact that there is no point R *on the paper* for which $RA=r-a$, $RB=r-b$, $RC=r-c$, when $r=a+b+c$, except in the special case when $\triangle ABC$ is a right-angled triangle.

The " solution " $r=a+b+c$ is not in general a solution of the *plane* problem which led to the equation satisfied, but is a solution of the problem in three dimensions,

" Three spheres whose centres are A, B, C and radii a, b, c touch each other externally, and also touch internally a sphere, centre R and radius r. Find r, if the sum of the areas of two faces of the tetrahedron $ABCR$ is equal to the sum of the areas of the other two faces."

It is interesting to consider the correct solution of the problem to find the radius r for the plane figure.

Let $\angle BRC = 2\alpha$, $\angle CRA = 2\beta$, $\angle ARB = 2\gamma$.

Then for all positions of R in the plane ABC, we have

$$\sin^2\alpha = \sin^2(\beta \pm \gamma).$$

If we expand this equation, substitute for $\sin \alpha$, etc., by the half-angle formulae for the triangles RBC, etc., and rationalize, we get a quadratic equation of which the solutions are

$$r = \frac{abc}{-(bc+ca+ab) \pm 2\sqrt{\{abc(a+b+c)\}}}.$$

The positive root is the required radius, and the negative root is interpreted as minus the radius of the circle which has external contact with the three given circles.

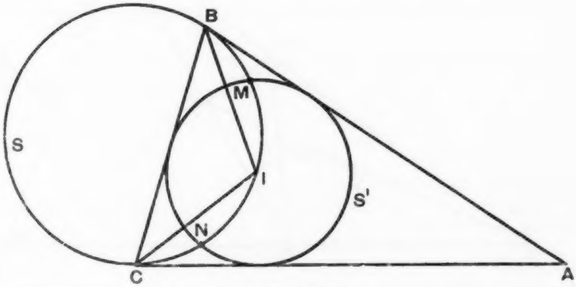
It should be noted that the equation obtained by considering angles is true only for points R of the plane ABC, and thus guarantees that the solution obtained is for a point on the plane. The equation obtained by consideration of areas can be true for points *not* on the plane ABC, and thus gives no such guarantee.

This note illustrates the remarkable fact, that a perfectly good solution of a perfectly true equation may not be the solution of the question being attempted, while it is the solution of some entirely different question. It seems that checking the authenticity of solutions is more important than is generally realised.

Richard Beetham.

2326. *Every conic is degenerate.*

Let S be a given conic. Take two arbitrary points B, C upon it, and let the tangents at B, C meet in A. Draw any conic, S', touching BC, CA, AB so as to meet S in four distinct points L, M, N, P.



Projected figure

Project L and P into the circular points at infinity. Then S becomes a circle with AB, AC as tangents, and S' the incircle of the triangle ABC.

Suppose that I is the middle point of the minor arc BC of S. Then, by elementary euclidean geometry,

$$\angle ABI = \angle BCI = \angle CBI = \angle ACI.$$

Hence I is the incentre of the triangle ABC; that is, I is the centre of S', so that *the centre of S' lies on S.*

In the original figure, therefore, *the tangent to S' at L meets the tangent to S' at P at a point on S.* Similarly, by projecting M, P and N, P in turn into the circular points, we may prove that the tangent at P to S' meets the tangents at M, N to S' at points on S. The tangent at P to S' therefore contains four points of S, so that S is degenerate, containing that line as part.

[On checking the examples for my book on homogeneous coordinates, Dr. J. A. Todd pointed out a fallacy in a question taken from the Preliminary Examination at Cambridge. In this note I pursue the fallacy to its logical conclusion.]

E. A. Maxwell.

2327. *On the centre locus of a four-line system of conics.*

That the centre locus of a four-line system of conics is a straight line is well known, but the identification of the type of conic which has a given centre on the straight line is perhaps not so widely appreciated. The ability to determine whether a conic with a given centre is an ellipse, hyperbola or parabola must have at least an aesthetic value for the student. The result may be stated thus :

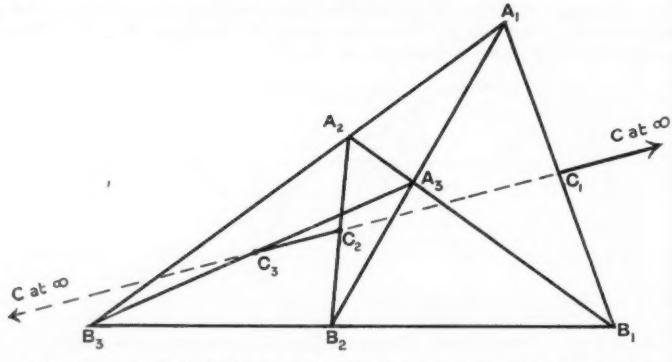
The straight line which is the centre locus of a four-line system of conics may be divided into four sections by the three centres of the point pairs of the system, and the centre (at infinity) of the single parabola. Two of the sections are the locus of the centres of hyperbolae, in general, and the two types occur alternately.

Denote a general four-line system of conics by

$$S \equiv \Sigma + \lambda\Sigma' = 0, \quad \text{.......................(i)}$$

where $\qquad \Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm,$

and $\qquad \Sigma' \equiv A'l^2 + B'm^2 + C'n^2.$



The conics touch the lines $A_1A_2B_3$, $A_2A_3B_1$, $A_1A_3B_2$, $B_1B_2B_3$. The three point pairs are A_1, B_1; A_2, B_2; A_3, B_3; and their respective centres ar C_1, C_2, C_3. The centre of the single parabola is C (at infinity). From C to C_1, we have centres of ellipses, and from C_2 to C_3; from C_1 to C_2 and from C_3 to C, centres of hyperbolae.

Taking (0, 0, 1) as the line at infinity, the centre of $S=0$ becomes

$$Gl+Fm+(C+\lambda C')n=0. \qquad \text{.........................(ii)}$$

The tangents from the centre of $S=0$ to $S=0$ are given by eliminating n from (i) and (ii), and the result, provided $C+\lambda C'\neq 0$, is easily found to be

$$l^2\{(A+\lambda A')(C+\lambda C')-G^2\}+2lm\{H(C+\lambda C')-GF\}$$
$$+m^2\{(B+\lambda B')(C+\lambda C')-F^2\}=0.$$

Thus the tangents from the centre of $S=0$ to $S=0$ are real, coincident or complex according as

$$\{H(C+\lambda C')-GF\}^2-\{(A+\lambda A')(C+\lambda C')-G^2\}\{(B+\lambda B')(C+\lambda C')-F^2\}\gtrless 0$$

and this condition can be reduced, either by direct calculation or by using the Laplace determinantal expansion, to

$$-(C+\lambda C')\Delta \gtreqless 0,$$

where

$$\Delta\equiv\begin{vmatrix} A+\lambda A' & H & G \\ H & B+\lambda B' & F \\ G & F & C+\lambda C' \end{vmatrix},$$

Now $\Delta=0$ is the condition for $S=0$ to degenerate to a point pair, so that

$$-(C+\lambda C')\Delta$$

changes sign, that is, we change from ellipses to hyperbolae or *vice versa*, when λ is such that $S=0$ becomes a point pair. Of course, it is only in general that

$$-(C+\lambda C')\Delta$$

changes sign at the roots of $\Delta=0$, that is, when the roots of $\Delta=0$ are distinct.

We have avoided discussing what happens when $C+\lambda C'=0$ since we have previously made the proviso that $C+\lambda C'\neq 0$. We overcome this difficulty as follows. From equation (ii) the centre of $S=0$ in cartesian coordinates is

$$\{G/(C+\lambda C'),\quad F/(C+\lambda C')\}.$$

So when $C+\lambda C'=0$ we have a parabola, its centre being at infinity. Now the tangents from the centre of a parabola to the parabola are coincident, so the presence of the factor $(C+\lambda C')$ in the expression $-(C+\lambda C')\Delta$ is justified. Therefore, in general the conics change their type at the roots of

$$-(C+\lambda C')\Delta=0,$$

that is, at the centres of the three point pairs, and at the centre of the one parabola. The centre locus in cartesian coordinates is

$$x=G/(C+\lambda C'),\quad y=F/(C+\lambda C'),$$

or, eliminating λ,

$$yG=xF.$$

Still one difficulty remains; we cannot tell which two sections of the centre locus are to be associated with ellipses and which two with hyperbolae, although we know that they occur alternately. This is not an altogether unexpected difficulty, since the question is one relating more to the figure than to our analysis.

F. A. BOSTOCK.

2328. *A property of the triangle.*

In a recent note in the *Gazette** Mr. E. J. Hopkins proved the following:

* Vol. 34 (1950), note 2144, pp. 129–133.

THEOREM. *If X, Y, Z are any three given points in the plane of a given triangle ABC such that*

$$\widehat{YAC}=\widehat{ZAB},\quad \widehat{ZBA}=\widehat{XBC},\quad \widehat{XCB}=\widehat{YCA}, \qquad \text{.................(1)}$$

then the lines AX, BY, CZ concur.

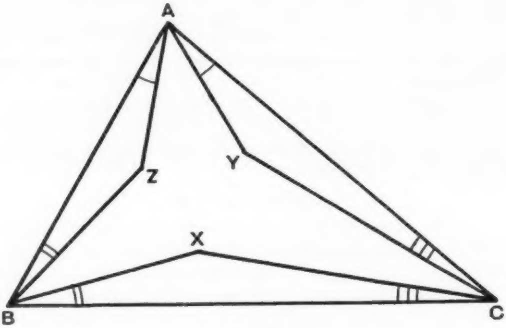


FIG. 1.

It may be of interest to compare the proof given by Hopkins with the following alternative derivation (which I had found independently). This depends on a modified form of Ceva's theorem, which we will state as a

LEMMA. *If lines are drawn through the vertices A, B, C of a given triangle so as to make angles θ, ϕ, ψ with AB, BC, CA respectively, then a necessary and sufficient condition for the lines to concur is*

$$\sin\theta\sin\phi\sin\psi=\sin(A-\theta)\sin(B-\phi)\sin(C-\psi).^{*} \qquad \text{............(2)}$$

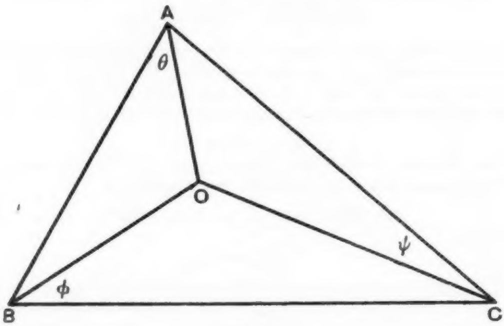


FIG. 2.

* We use here the convention that θ is to be measured in the positive (anti-clockwise) sense about A from AB (though the line through A may well lie outside the triangle, and θ need not be restricted to lie in the range $0\leqslant\theta<2\pi$), and similarly for ϕ, ψ. The *senses* of the lines are, of course, irrelevant, since (2) is unaltered if we increase any of the angles θ, ϕ, ψ by π (or, indeed, by any multiple of π), while the condition of concurrency is also unaffected.

Proof of Lemma. We shall suppose for the necessity part of the proof that the point O of concurrency does not coincide with any of A, B, C, and, for the sufficiency, that none of θ, ϕ, ψ is a multiple of π. However, it is easy to see that the result holds trivially in these degenerate cases, and therefore completely generally (for any given fixed determinations of θ, ϕ, ψ), provided that we regard the case in which the lines are *parallel* as a case of concurrency.

If the lines intersect in a point O, then we have

$$\frac{\sin(C-\psi)}{\sin\phi}=\frac{BO}{CO}$$

and two similar relations, so that

$$\frac{\sin(C-\psi)}{\sin\phi}\,\frac{\sin(A-\theta)}{\sin\psi}\,\frac{\sin(B-\phi)}{\sin\theta}=\frac{BO}{CO}\,\frac{CO}{AO}\,\frac{AO}{BO}=1,$$

i.e. the condition (2) is *necessary.*

Conversely, if we regard A, B, C, θ, ϕ as fixed, then, since

$$\frac{\sin(C-\psi)}{\sin\psi}=\sin C\cot\psi-\cos C$$

is a strictly decreasing continuous function of ψ in any fixed range

$$k\pi<\psi<(k+1)\pi,$$

and tends to $+\infty$, $-\infty$ as ψ tends to $k\pi$, $(k+1)\pi$ respectively from within the range, it follows that there is precisely one value of ψ in the range for which (2) holds. Hence, by what we have already proved, we see that, if (2) holds, then the lines must concur, *i.e.* the condition is also *sufficient,* as required.

Proof of Theorem. Let the angles in (1) above be denoted α, β, γ respectively, and write

$$X\widehat{AB}=\theta,\quad Y\widehat{BC}=\phi,\quad Z\widehat{CA}=\psi.$$

Then clearly

$$\frac{\sin\theta}{\sin(B-\beta)}=\frac{BX}{AX},$$

and similarly

$$\frac{\sin(A-\theta)}{\sin(C-\gamma)}=\frac{CX}{AX},$$

so that

$$\Pi(\sin\theta)=\Pi\frac{\sin(B-\beta)}{AX}\,\Pi BX,$$

$$\Pi\sin(A-\theta)=\Pi\frac{\sin(C-\gamma)}{AX}\,\Pi CX;$$

hence, by the Lemma, it will be enough to prove that

$$BX\,.\,CY\,.\,AZ=CX\,.\,AY\,.\,BZ.$$

But

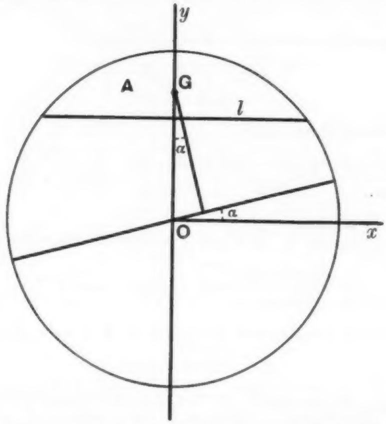
$$\frac{BX}{CX}=\frac{\sin\gamma}{\sin\beta},$$

and similarly, so the theorem follows.

M. P. Drazin.

2329 *A note on mensuration.*

With a diameter as axis, a tapered hole is bored through a sphere. If the hole is in the form of the frustum of a right circular cone of semi-vertical angle α, from which the sphere cuts off segments of its generators of length $2l$, the volume remaining is $\frac{4}{3}\pi l^3 \cos \alpha$.



1. A segment of area A, of a circle of radius a, cut off by a chord of length $2l$, rotates through 2π about a diameter parallel to the bounding chord. Suppose that the centroid G of the segment is distant $\bar{y}$ from the diameter. Then the volume swept out is

$$2\pi \int_0^l (a^2 - x^2)\,dx + 2\pi l(l^2 - a^2).$$

Thus
$$2\pi A\bar{y} = \pi(2a^2 l - \tfrac{2}{3}l^3 + 2l^3 - 2a^2 l) = \tfrac{4}{3}\pi l^3.$$

2. Suppose that this segment now rotates through 2π about a diameter inclined at α to the bounding chord. The volume swept out is

$$2\pi \,.\, A \,.\, \bar{y} \,.\, \cos\alpha = \tfrac{4}{3}\pi l^3 \cos\alpha.$$

A. J. L. AVERY.

2330 *A query.*

Copies of text-books one used at school have interests comparable with those of old letters and diaries ; mine show many problems marked by ? (failure at first attempt) and ?? (second defeat) ; ??? stands at the side of the following example in Edwards' *Differential Calculus* :

If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3 the lengths of the three tangents drawn from any point to the semi-cubical parabola $ay^2 = x^3$, then will

$$27 n_1 n_2 n_3 n_4 = a t_1 t_2 t_3. \qquad \text{(Math. Tripos, 1890)}$$

Formation of the quartic whose roots are the n's and of the cubic whose roots are the t's is not outrageously laborious, and verification of the result is then immediate. But is there a *short* proof of this property, and, if not, how was the result evolved?

C. V. DURELL.

2331. *A simplification of Bertrand's method for integrating a total differential equation.*

F. Underwood (*Math. Gaz.*, Vol. XVII, 1933, p. 111) has pointed out that of the four general methods for integrating the total differential equation

$$E \equiv P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz, \quad \text{.................(1)}$$

supposed completely integrable, Mayer's method requires one integration, the most usual method (which starts by treating one variable as constant) and the use of an integrating factor each require two integrations, while Bertrand's method requires no less than three. Moreover, of these three, two may, in general, require some artifice for which no definite rules can be given. The object of this note is to show how we can dispense with one of these two fortuitous operations, and replace it by the routine integration of a perfect differential.

Bertrand's method is explained in several treatises, *e.g.* Forsyth's *Treatise on Differential Equations*, 6th ed. 1929, pp. 331–2, or Ince's *Ordinary Differential Equations*, 1927, pp. 59–60. We start by finding two independent integrals, say $\alpha(x, y, z) = a$, $\beta(x, y, z) = b$, of the simultaneous equations

$$dx \Big/ \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = dy \Big/ \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = dz \Big/ \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right). \quad \text{............(2)}$$

Equation (1) can then be reduced to the form

$$E \equiv A(\alpha, \beta)\,d\alpha + B(\alpha, \beta)\,d\beta = 0, \quad \text{..............................(3)}$$

involving only two variables α and β. After integration of equation (3), α and β are replaced by their expressions in terms of x, y, z.

Both Forsyth and Ince introduce Bertrand's method as a sequel to the reduction of E to the canonical form $du + v\,dw$ and both mention that the condition of integrability ensures that each of u, v, w is a function of α and β alone. Yet no one seems to have noticed that this makes it unnecessary to determine more than one integral directly from the equations (2). Call this integral $w(x, y, z) = a$. Applying the method given by Forsyth, p. 329, or Ince, p. 58, for the case when $E = 0$ is not completely integrable, use the relation $w = a$ to express one variable in terms of the other two and of a, *e.g.* $z = f(x, y, a)$. By substituting for z in E, we shall reduce it to a perfect differential, say

$$d\phi\{x, y, a\},$$

and u can be taken as $\phi\{x, y, w(x, y, z)\}$. Then v can be found from

$$R = \frac{\partial u}{\partial z} + v\frac{\partial w}{\partial z}.$$

This v is a function of u and w, so $E = 0$ is reduced to the form

$$du + F(u, w)\,dw = 0.$$

We integrate this and then replace u and w by their values in terms of x, y, z.

Comparing this with Bertrand's method, w is the same as α, but v is a special form of β which reduces $A(\alpha, \beta)$ to 1, and moreover is determined by a definite "fool-proof" process, requiring no artifice or guesswork.

If we apply this new method to the example that Forsyth (p. 332) works by Bertrand's method, *viz.* $E \equiv -zy\,dx + zx\,dy + y^2\,dz = 0$, the simultaneous equations (2) becomes $dx/(x - 2y) = dy/y = dz/(-2z)$, of which $w \equiv y^2 z = a$ is one integral. Putting $z = a/y^2$ in E, we get

$$-\frac{a\,dx}{y} + \frac{ax\,dy}{y^2} - \frac{2a\,dy}{y} = d\left(-\frac{ax}{y} - 2a\log y\right),$$

so $$u = -xyz - 2y^2z \log y,$$
and $$y^2 = (-xy - 2y^2 \log y) + vy^2,$$
so $$v = 1 + \frac{x}{y} + 2 \log y = 1 - \frac{u}{w}.$$
Thus $$E = du + \left(1 - \frac{u}{w}\right) dw = 0,$$
giving $$\frac{u}{w} + \log w = c, \quad \text{or} \quad -\frac{x}{y} + \log z = c.$$

But this example is not a good one to illustrate the merits of either Bertrand's method or my simplification, as the equation is homogeneous and so integrable at once, after multiplication by the integrating factor $1/(Px + Qy + Rz) = 1/y^2z$.

A better example, not unduly easy by Bertrand's or any other standard method, is

$$E \equiv (-2xz + x^2y + yz^2 - y^4)dx + (xz^2 - 3y^2z - x^3 + 2xy^3)dy + (x^2 - x^2y^2 + y^3)dz.$$

The simultaneous equations (2) become

$$dx/(2xz - 6y^2 + 2x^2y) = dy/(4x - 2yz - 2xy^2) = dz/(4x^2 - 6y^3).$$

Each of these ratios equals $(y\,dx + x\,dy)/(4x^2 - 6y^3)$, so $dz = y\,dx + x\,dy$ and $w \equiv z - xy = a$ is one integral. Putting $z = xy + a$ in E, we get

$$(-2ax + 2axy^2 + a^2y)dx + (-3ay^2 + 2ax^2y + a^2x)dy$$
$$= d\{a(-x^2 + x^2y^2 - y^3) + a^2xy\},$$
so $$u = (z - xy)(zxy - x^2 - y^3),$$
and $$x^2 - x^2y^2 + y^3 = (2zxy - x^2 - x^2y^2 - y^3) + v,$$
so $$v = 2(x^2 + y^3 - zxy) = -2u/w.$$

Thus $E = du - 2u\,dw/w = 0$, giving $u = cw^2$, or $zxy - x^2 - y^3 = c(z - xy)$.

H. T. H. Piaggio.

2332. *On Note* 2144 : *theorems on concurrence and collinearity.*

An interesting alternative proof of the main result (IV) of this note may be obtained by using Brianchon's theorem.

Since the diagonals of the hexagon $X_1Y_2Z_1X_2Y_1Z_2$ are concurrent, there is a conic inscribed in the hexagon. The hexagons $XY_1ZX_1YZ_1$ and $XY_2ZX_2YZ_2$ each consist of the same six tangents in a different order and so the diagonals of each are concurrent. The theorem that the points of concurrence are collinear is the dual of part of Steiner's theorem on Pascal lines (see, for example, Todd, *Projective and analytical geometry*, pp. 82—84).

This theorem also shows that there are three other hexagons, $ZY_2Z_1YZ_2Y_1$, $XZ_2X_1ZX_2Z_1$, $YX_2Y_1XY_2X_1$, whose diagonals are concurrent, and that the line joining the points of concurrence for the first set of three hexagons is conjugate (with respect to the conic) to the line joining the points of concurrence for the second set.

E. J. F. Primrose.

2333. *An enigma.*

A candidate, answering a question on factors recently, wrote :

" $2x^2 - x - 1 = (2x + 1)(x - 1)$ by F.M.O.L."

Can anyone explain the capital letters?

C. O. T.

REVIEWS

Methods of Algebraic Geometry, II. By W. V. D. Hodge and D. Pedoe. Pp. ix, 394. 42s. 1952. (Cambridge University Press)

The considerable developments in algebraic geometry in the last twenty-five years have not yet percolated through to the textbooks of the subject. Indeed, the assimilation of this new material is not yet by any means complete, and the attempt to write a truly modern textbook on algebraic geometry requires a certain degree of courage as well as mastery of the subject. The authors of the book under review deserve our admiration for making the venture, no less than our congratulations on the success with which they accomplish it. This second volume of the series enables one to see more clearly the plan of the whole work, although a final appreciation must wait until the appearance of the third and final volume. Volume I is now seen to be mainly preparatory, and some of the material such as, for instance, Grassman spaces, which appeared to have undue prominence in that volume, now falls into perspective.

The work naturally owes much to other writers, and in this volume the influence of van der Waerden is particularly apparent. The reader will also notice the affinity with A. Weil's *Foundations of Algebraic Geometry*. Those interested in the sources can have no better guide than the authors' own Bibliographical Notes. The book is however in no sense a summary of, or even an introduction to, recent original work. The authors' avowed intention is an exposition of method, and not an account of results. Perhaps it would be incorrect to describe the approach as new, but a systematic development of the foundations of the subject on these lines has not before been given in a form accessible to the general reader.

This volume is divided into two parts, Books III and IV. Book III deals with fundamental theory. It begins with a discussion of algebraic varieties; ground fields of finite characteristic are wisely excluded. The treatment rests on the *fundamental form*, or *Cayley form*, of a variety. For a variety of dimension d in the ambient space S_n, this form may be thought of as representing the cone containing the given variety and having a generic S_{n-d-2} as vertex. The vanishing of the form expresses the condition that a generic S_{n-d-1} meets the variety. The great advantage of using the Cayley form is that a variety is thereby represented by a single algebraic form. After a discussion of irreducible varieties (for which the Cayley form is irreducible), a *variety in the multiplicative sense* is defined as a set of irreducible varieties of like dimension with multiplicities attached. Such varieties correspond in an obvious way to power products of the Cayley forms of the irreducible components. These "varieties in the multiplicative sense" are essentially the *cycles* of A. Weil, and it might have been better to adopt the latter term. They are of course logically distinct from the *varieties* originally defined by the authors as sets of points, and the distinction is made clear throughout the book by using Clarendon symbols for cycles. This Book contains a discussion of multiplicity on the basis of relation-true specialisation, followed by a careful discussion of intersection theory, and concludes with some remarks on equivalence and the theory of the base. Much of the material of this Book has appeared before, e.g. in van der Waerden's *Algebraische Geometrie*, but the account given here is more detailed and complete than those hitherto available, and forms a valuable addition to the expository literature.

Book IV is concerned with applications ; the two chapters deal with Quadrics and Grassmann varieties. The first of these contains much that is a digression from the main course of the book, and one is inclined to wish that certain topics, such as self-transformations of a quadric and simultaneous reduction

of two quadratic forms, had been safely disposed of in Volume I. The advantage of getting the preliminary work cleared out of the way is seen in the very pleasing final chapter, in which the topics discussed are all of great interest and importance. This chapter includes the theorem that the Schubert varieties of a given dimension are a base for varieties of that dimension on the Grassmannian, explores some applications to enumerate geometry, and concludes by establishing the postulation formula for a Schubert variety.

E. C. T.

Mécanique des Milieux Continus et Déformables. I. II. By M. Roy. Pp. xxii, 362; xii, 350. 5325 fr. 1950. (Gauthier-Villars)

The tradition in British universities is to treat the various branches of the mechanics of continuous media as separate subjects. A good deal of hydrostatics is learnt before hydrodynamics is begun, and as often as not no elasticity beyond simple beam theory is attempted. This is unsatisfactory but it is a course which is dictated by the intellectual capacity of the average undergraduate. No one who has ever taught the mechanics of continua doubts that the ideal method is to treat the subject as a whole, to complete the analysis of strain and stress in a general deformable body before discussing hydromechanics or the mechanics of deformable solids. Such a course does however impose a very severe burden on an undergraduate, and it is usually found more practicable to discuss some elementary parts of hydromechanics and elasticity before attempting the general theory which demands a reasonable acquaintance with the simpler properties of tensors and a firm understanding of the principles of thermodynamics. The student usually has this equipment at his disposal only towards the end of his undergraduate course, but any attempt to present the subject to undergraduates in a comprehensive way is sure to be received with interest by teachers of applied mathematics.

This book by Professor Roy is such an attempt. It tries to provide, in a form suitable for engineers, a rigorous exposition of the fundamental principles of the mechanics of continua and of the mathematical techniques necessary for the solution of special problems. The author is to be admired for his courage in undertaking alone the writing of a work covering such a vast field, and to be congratulated on the skill with which he has accomplished his heavy task. He has certainly achieved his aim of showing the inter-relation between the theories of hydrodynamics and of elasticity in a concise and remarkably lucid manner.

The two volumes contain just over four hundred pages of text and just under three hundred pages of appendices which amplify the discussions in the main body of the book, and the sections of which are so numbered as to make cross reference easy. Each volume has two main subdivisions. The general theory of the deformation of continuous media and the theory of elasticity make up the first volume; the second is devoted to hydrodynamics and theory of machines.

Part I is perhaps the most interesting section of the book to a teacher of the subject. The author bases his unified treatment of the mechanics of continua upon the incorporation of the concepts of thermo-dynamics into classical mechanics, a natural method of allowing for exchanges of heat and variations of the physical or chemical state if we wish to avoid considerations of the structure of matter. The first chapter begins with a brief but clear outline of Cauchy's analysis of the strain in a continuous medium, introducing the student to the ideas of the strain tensor and the strain quadric. The kinematics of such a medium is then discussed by both the Lagrangian and the Eulerian method. The whole chapter is geometrical in character in the sense that it does not consider the cause of the strain in the body, and would be

likely to prove difficult to a student of engineering. The second chapter, which gives an account of the principles and the fundamental equation of thermodynamics, is probably the most difficult in the whole book. It is an admirable account of the foundations of the subject to anyone who already has a pretty thorough grasp of it and, as such, it will be of value to applied mathematicians, but a student (especially an engineer) will get more value from the less rigorous physical approach common to elementary text books of thermodynamics. The second law, for instance, is given in Jouget's form, which should appeal to mathematicians, but whatever mathematicians may think, the engineer still prefers to build up his thermo-dynamical concepts in terms of perfect gas engines performing Carnot cycles. There then follows a very short section on general theorems in mechanics. The fourth chapter applies these general ideas to deformable media as a whole, and then more particularly to elastic solids. It treats such topics as the definition of the internal state of a medium by temperature and deformation, the distinction between solids and liquids, the stress tensor, and (in appendices) the Mohr circle and the stress ellipsoid. In the specialisation to elastic solids a very full discussion of the stress-strain relation is given. In the last chapter of this part these ideas are extended to viscous and inviscid fluids, the point of departure being the thermodynamic definition of a fluid. Navier's hypothesis, the dissipative function and the Helmholtz Free Energy of a fluid are treated fully before the equations of continuity and motion are derived. At the end of this chapter there is an illuminating discussion of the hypotheses upon which fluid mechanics rests and of similarity principles in this subject. The hundred and fifteen pages which comprise Part I may, as has been suggested, prove to be difficult reading for a student; they will most certainly prove to be stimulating and interesting to anyone charged with lecturing on the topics discussed in them.

In Part II we are on more familiar ground in the sense that it treats a subject—the mathematical theory of elasticity—on which a few excellent text-books already exist. The first chapter is concerned with general matters such as the equations of equilibrium and compatibility, the superposition principle, the idea of elastic equilibrium and Castigliano's theorem. The corresponding part of the appendix deals adequately with phenomena near the elastic limit. The first applications of these general concepts to particular problems occur in Chapter II which deals with plane strain and plane stress in an obvious way. The Airy stress function is introduced to solve the equations in Cartesian co-ordinates and some discussion is given of the plane biharmonic equation to which this type of solution leads. The use of complex variables is described briefly but no mention is made of the work of A. C. Stevenson and A. E. Green which increased so considerably the power of this method. An appendix contains a short account of the principles of photoelasticity. Chapter III deals with elastic problems in three dimensions and here again the choice of problems treated is a little unimaginative—the bending of prisms, torsion, St. Venant's problem, approximate beam theory etc. There is also a short but good discussion of the buckling and stability of thin rods. Part II ends with a chapter on the small vibrations, about an equilibrium state, of elastic bodies in which the general theory is developed and then applied to vibrations of thin bars.

Of the four parts, the third, which is devoted to hydrostatics and hydrodynamics, is the longest. It contains to some degree most of the material to be found in standard university texts, although in the discussion of particular problems the mathematical techniques used here are less sophisticated than in most other textbooks of this kind. The first chapter is on hydrostatics; in addition to the general theory it covers topics such as the isothermal atmosphere and the stability of floating bodies.

The next four chapters form a set given over to the dynamics of inviscid, incompressible fluids. The first of the group sets up the equations of motion and the general theorems associated with the names of Lagrange, Helmholtz, Bernoulli and Torricelli derivable from them. Plane irrotational flow is considered in more detail in the next chapter. The complex potential is the centre of this discussion which ranges over the theory of conformal representation, the Blasius formulae and aerofoil theory. Numerous special problems are treated in appendices. In the next two chapters we proceed to three-dimensional problems which are solved by the standard methods—sources, doublets, images etc., and finally to a discussion of three-dimensional vortex motion with applications to Prandtl's wing theory.

There follows an account of wave propagation in fluids which are compressible and of the properties of shock waves. Special topics considered are Tchapligyn's approximation, supersonic flow past a wedge and an ogival head, the construction of flow fields by the use of characteristics, and the linearisation of the equations of motion.

The last chapter of Part III on viscous flow is much less detailed than its predecessors. It treats laminar flow and turbulent flow briefly, the Poiseuille formula for flow in a pipe being the only special result examined in any detail. The theory of Millikan's oil drop experiment may be more appropriate as an illustration in lectures to students of physics but it seems too good to be omitted from a work of this kind. The chapter ends with a discussion of the principles of hydraulics, treating as particular examples flow through nozzles, and the Borda mouthpiece.

The last part of the book is of little interest to mathematicians. Treating machines which consist of solid and fluid parts which interact with each other only through the action of forces or the exchange of heat it is meant to serve as an introduction to more specialised engineering monographs. The first of the two chapters comprising it deals with turbomachinery, while the second discusses briefly the theory of variable speed machines.

While, on the whole, the author has achieved a satisfactory synthesis of the subject, the book is not without some defects. It is rather surprising to find that although some of the basic ideas which it presents, say in the first part, would prove difficult to an engineer, this work does not give any account of some of the most powerful mathematical tools available for the solution of special problems. It is right to insist that the engineer should understand fully the foundations of the subject, but if he is going to do research or development work he may have to solve boundary value problems of some complexity. Professor Roy's present book will not equip him to do that ; it makes no mention even of the standard procedures available for the solution of the second order partial differential equations which arise frequently in this field. A good account is given of the method of conformal representation, but spherical harmonics, Bessel functions, Fourier series, and integral transforms do not receive a mention. These techniques can be used to solve significant practical problems and it is hard to believe that a student would find their use any more difficult than he would the theoretical concepts developed in the earlier chapters.

The division of the book into a main text and a large number of appendices has not been an entirely happy one. This method is often most effective—as Max Born has shown in recent years—but it must be used consistently. Its main purpose should be to ensure that the reading of the text is not rendered difficult by the inclusion of too much (if any!) mathematical detail. The author of the present book does not seem to have any rational scheme ; sometimes appendices contain only special examples of the use of general theory ; and that is just as it should be. At other times quite important pieces of

general theory (as, for instance, the whole discussion of group velocity) seem to have been put into appendices instead of into the text itself. Once in an appendix it is possible that they will remain unread by the student, who will naturally assume that important results are not dealt with in this way.

The main danger is that the class of student for whom it was written will not be sufficiently mature to appreciate the earlier parts of this work. This does not, however, diminish the great service to the subject the author has done by insisting on a proper treatment of its fundamentals in what he frankly describes as a class text. These two volumes will certainly be of great value to those engaged in training students in applied mathematics, and form a most reliable and excellent reference book for research workers in that or allied fields.

I. N. S.

Théorie Mathématique du Risque dans les Assurances de Répartition. Vol. I. By J. Doubourdieu. Pp. 306. 3500 fr. 1952. (Gauthier-Villars, Paris)

At intervals from 1937 onwards Gauthier-Villars has published a series of monographs on the calculus of probability and its applications under the general editorship of the distinguished French mathematician, M. Emile Borel. The present treatise is the eighth of the series and forms the first of two volumes on an aspect of probability which, as its title denotes, is of special interest to actuaries.

In the early chapters of the book the author surveys the fundamental principles on which the application of the calculus of probability to actuarial and similar problems rests. He then proceeds to elaborate the classical and modern theories of risk in relation to life and other forms of insurance. Much of this is highly specialised and will probably have little appeal to the pure mathematician, except perhaps for chapter V where the theory of risk is applied to the problem of what is picturesquely termed " la ruine des joueurs".

The book is well written and is worthy of study by the reader who is interested in the mathematical side of actuarial science. M. Dubourdieu is to be congratulated on having produced a volume worthy of inclusion in this important series of mathematical treatises.

H. F.

Memorie Scelte. By P. Burgatti. Pp. vi, 354. 2500 lire. 1951. (Zanichelli, Bologna)

This volume of selected papers by the late Professor Pietro Burgatti, who died in 1938, has been published under the auspices of the Universities of Bologna and Ferrara, the Academy of Science of Bologna and the Unione Matematica Italiana. It contains thirty-eight papers, one third of the total number published by the author. The selection represents fairly well the interests of Burgatti, which ranged over the fields of Pure and Applied Mathematics. The selection made includes three papers on Geometry, five on Differential Equations, two on Integral Equations, nine on Vector Analysis, ten on Classical Mechanics, five on Elasticity, two on Astronomy and two on Relativity.

Only a few of these papers can be mentioned here, but it is hoped that this will give sufficient indication of the types of problem in which the author was interested. There is, for instance, an interesting paper on the extension of Riemann's method of solution of the hyperbolic partial differential equation of the second order to equations of the type

$$\sum_{s=0}^{n} a_0 \frac{\partial^n z}{\partial x^{n-s}\partial y^s} + \sum_{s=0}^{n-1} a_1 \frac{\partial^{n-1} z}{\partial x^{n-1-s}\partial y^s} + \dots + a_n z = 0,$$

where the a's are functions of x and y. In a paper on Integral Equations the author develops a method of successive approximations for equations of the type

$$\phi(y)=\int_0^y\left(\sum_{r=0}^{n}\psi_r(x, y)\frac{d^{n-r}f(x)}{dx^{n-r}}\right)dx$$

which is simple, elegant and avoids the use of determinants. There can be little doubt that the problem of the motion of an asymmetrical top about a fixed point was one of absorbing interest to the author. One such problem may be mentioned here—the determination of first integrals of the form

$$f(p, q, r, a, b, c)=\text{constant}$$

(where p, q, r, are the resolutes of the angular velocity about the principal axes at the fixed point and a, b, c are parameters) in the special case in which one of the six quantities, p, q, r, ... c, is explicitly absent. Burgatti derives afresh the well-known results associated with the names of Liouville, Lagrange and Kowalevski and furthermore shows that there are no first integrals which are algebraic or transcendental which depend on *more than five* of the six variables—an interesting and important result. In another paper on mechanics he extends a theorem due to Staeckel relating to the conditions under which a conservative dynamical system of n degrees of freedom should possess, in addition to the energy integral, $n-1$ similar integrals quadratic in the generalised velocities.

Of the several papers on elasticity, one of special interest relates to the theorem that the displacement of a point of an isotropic elastic solid in equilibrium under surface forces can be expressed in terms of three harmonic vector functions. In another paper the author uses this result to solve known problems associated with the names of Clebsch and Saint-Venant.

The author does not appear to have been quite so successful in the papers on Astronomy and Relativity. In a note on the theory of comets the author reconsiders the contradictory results of Laplace and Schiapparelli on the probability of hyperbolic orbits and shows that these two points of view can, in fact, be reconciled. He also shows that there is no contradiction involved between the fact that cometary orbits are elliptic even at large distances from the Sun and the hypothesis that these orbits arise in the combined gravitational fields of the Sun *and distant stars.* This theory, though interesting and ingenious, does not seem very convincing. Of the two papers on Relativity, one deals with the Lorentz Transformation as a homography in an S_4 and the other with the advance of perihelion of planetary orbits. In this the author points out that the classical solution of this problem, based on spherical symmetry of the central mass, does not allow for the asymmetry due to the flattening of the Sun's core produced by its rapid rotation. Burgatti claims that this point is important and, in fact, the results appear to conflict with observations even if the flattening is small. (The result is based on work of Vogt who maintained that the Sun's core is in fast rotation and consequently highly flattened, but here also the result seems highly speculative.)

There is an interesting paper (a seminar given in Moscow in 1934) in which the author considers the relative merits of the intrinsic general vector calculus developed by Burali-Forti, Marcolongo and Boggio and the Tensor Calculus, which, the author says, relies on coordinates to establish an algorithm capable of translating in analytical form intrinsic properties of space. Burgatti favours the vector calculus and gives a simple example in support of his thesis. In this he points out that whilst the tensor calculus method of solution requires long and artificial methods, the general vector calculus gives a more direct and " expressive " solution.

Altogether, this is a very interesting collection of papers and one is grateful to the sponsors for having made available to the mathematical world in book form some of the work of this eminent mathematician.

V. C. A. FERRARO.

Exercises in elementary mathematics. IV. By K. B. SWAINE. Pp. 256, 16 (tables). 8s. 6d. Answers, 3s. 6d. Teachers Book, 3s. 6d. 1951. (Harrap).

This book completes the course the first three books of which have already been reviewed in the *Gazette*. It commences with a brief summary of the work of books 1–3 followed by a repetition of the last 10 pages of book 3. Book 4 is planned to cover the requirements of the last two years of a five-year course based on the Alternative Syllabus. It continues the plan of omitting explanations and leaving these to the teacher. Formal proofs of theorems are not given and the author suggests that teachers may prefer to give their own notes on these. On the other hand, the main facts are approached by series of graded questions. The cosine and its applications are introduced. Most teachers would prefer to deal with the cosine at the same time as the sine. In solid geometry there are a number of exercises on sections of well-known solids and these will be welcomed. On the whole, there is an adequate number of exercises on each topic and these usually cover a wide range of applications. Most of the topics conclude with a set of miscellaneous questions which are in the nature of revision. At the end of the book there are sets of revision examples, arranged under topics, which average about 12 per set. The book ends with 49 miscellaneous revision tests, each of five questions, thus making 157 tests for the four volumes, a goodly number.

Criticisms are offered on certain points. A number of topics which are not in the Alternative Syllabus and which occupy many pages of the book are dealt with at some length. There are the inequality theorems in geometry, arithmetical and geometric progressions in algebra, and rather more detailed manipulative work on indices than is needed. The work on the theory of quadratic equations is also outside the scope of a work of this kind. The extension of Pythagoras is now an anachronism which is displaced for all purposes by the cosine formula. As far as the average pupil is concerned, it is a waste of time besides being outside the Syllabus. Finally, there is the treatment of functionality. I think it was Sir Percy Nunn who stated that graphical treatment should be related to functionality and that the co-ordinate geometry approach is altogether wrong. The correctness of this view cannot be questioned. Nevertheless, the author's approach is entirely one of coordinate geometry.

S. I.

Higher Certificate and Intermediate tests in pure mathematics. By R. J. FULFORD. 3rd edition. Pp. 139. 3s. 9d. 1951. (University Tutorial Press)

Mr. Fulford's excellent collection of examples in pure mathematics for the top forms now appears in a third edition, somewhat extended to cover changes of syllabus. Differential equations (15 questions) are now included. A wide variety of well-chosen exercises has already made the booklet a favourite with many teachers.

T. A. A. B.

School arithmetic. By W. P. WORKMAN. 4th edition, revised by G. H. R. Newth. Pp. viii, 548. 8s.; without answers, 7s. 3d. 1951. (University Tutorial Press)

This issue of a fourth edition of a world-famous text calls for congratulation of the reviser upon an excellent piece of work. He has wisely retained in full the late Mr. Workman's admirable explanations and expositions of arith-

metical processes, and, where necessary, he has brought examples up to date by revising prices, and referring to recent developments, such as nationalisation, taxation and air travel. A few topics, such as cube roots and continued fractions, also dealt with in Mr. Workman's *Tutorial arithmetic*, have been omitted, and in their place a large collection of examination questions has been substituted.

From the point of view of the English grammar school, it seems a pity that the reviser has not been more drastic in his attitude to the order of treatment of topics in the latter half of the book. Thus logarithms are not introduced until after such topics as compound interest and mensuration have all been fully covered. The normal practice in English schools is to use logarithms quite freely in work on these topics—indeed they are essential for doing inverse problems in them.

It would also seem that too much room is devoted to a detailed treatment of contracted methods, whereas there is no recognition of the existence of tables of squares, square roots or reciprocals. The treatment of logarithms is not on the same scale as that of many more elementary topics, and no indication is given of how to set out a logarithmic calculation.

The publishers are to be congratulated on the clearness of the type used, and the skilful way in which type faces and sizes are combined in the layout of the pages.

F. J. T.

Calculus. By Tomlinson Fort. Pp. xii, 560. 25s. 1952. (Heath, Boston ; Harrap, London)

It is refreshing to encounter an author who holds and acts upon the belief that the proper way to present the subject of calculus to students of engineering is with mathematical rigour. In contrast to most text-books for engineers, Professor Fort's book places emphasis on the fundamental definitions and theorems of analysis rather than on drill in mathematical technique. The *Calculus Report* does not recommend such a rigorous treatment in a first course at schools, but it might well appeal to a more mature student making a later start, or serve as a useful bridge for scholarship candidates between school work and the course in analysis at the university. This book provides a readable introduction to the conceptions of precise definition and rigorous proof. Not all the theorems stated are proved and the development stops short of such difficulties as the continuity of the sum-function of a power series at the end points of its interval of convergence. There is one light-hearted explanation of an unproved theorem which should perhaps be supplemented by a diagram. Of the theorem that a continuous function must take all values between its end values we read : " it states that if we cannot jump, or fly, or tunnel, and if there is no bridge or boat or way around, it is impossible to get from one side of a river to the other side without getting wet."

The opening sequence is, in brief : convergence of series, limits and continuity, differentiation and some applications, mean value theorem. In Chap. 8 the Riemann integral is defined and the fundamental theorem connecting integral with derivative is accorded the dignity of constituting Chap. 9. In Chap. 12 the Napierian logarithm is defined as an integral. Attention is paid to the remainder after n terms of Taylor's series and, similarly, when approximate integration is reached, the mean value theorem for integrals is used to find upper bounds to the errors involved. In the chapter on improper integrals the integral test of convergence of series is proved. As a prelude to partial differentiation and multiple integrals there is a chapter on solid analytic geometry, and the book concludes with a chapter on differential equations taken as far as the second order equation with constant coefficients.

Large type and generous spacing contribute to the readability. The diagrams are simple and bold, though a few of them seem somewhat misleading. Fig. 121, illustrating the sum of the projections of consecutive straight line segments in three-dimensional space, in fact makes the segments coplanar; the cardioid in Fig. 93 and the companion limaçon in Fig. 98(*d*) both appear to have a branch point where the one should have a cusp and the other a simple point. There are some differences from our usual conventions. A left-handed system of axes is used in three dimensions, and in $dy/dx = \tan\alpha$, α is chosen as positive acute or obtuse. There is an obvious minor misprint in the last line of p. 193, the numerator of the index $\frac{1}{2}$ having been omitted. A serious misprint is in the list of integrals on p. 319, where $\tanh^{-1}x$ is given as the integral of $1/(x^2-1)$. The author is not in favour of giving answers but compromises to the extent of giving them to odd-numbered examples. It should be stated that the examples are in general easy.

It will be seen that as a text-book for engineering students, Professor Fort's book would hardly suit the requirements of English courses, in which the emphasis is strongly on the acquirement of skill in using calculus as a tool. Nevertheless the book should prove useful, as suggested, as an introduction to analysis, and teachers in schools and technical colleges might well like to have such a book for reference. C. G. P.

Advanced National Certificate Mathematics. By J. PEDOE. Pp. xiv, 346. 15s. 1952. (English Universities Press)

This volume in the English Universities Press Technical College Series is designed for students at the A1 level in Higher National Certificate courses and also for degree students at, for instance, the level of Part I of B.Sc.Eng. (London). As the title perhaps indicates, the former type of student is more particularly in mind; a full-time degree student should be able to take some topics rather further. In courses for part-time students the main objective is the acquisition of a fair mathematical technique over a rather wide range in a short time. Mr. Pedoe succeeds in covering the ground in a way which should prove neither alarming to the student, nor offensive to the mathematician.

It is stated that the book may be regarded as a sequel to Vol. III of *National Certificate Mathematics* by Mahon and Abbott. Of the 35 short chapters, some early ones are devoted to revision, in particular of the differentiation of e^{ax}, $\log x$ and the circular functions and the use of natural logarithms. Coordinate geometry and curve sketching occupy 10 chapters as a preliminary to integration and its applications, which include centres of gravity, moments of inertia, centres of pressure, linear motion of a particle and rotation of a rigid body about a fixed axis. There is a satisfactory chapter on convergence of series before Maclaurin's series is introduced; the more general Taylor's series is not included. The chapters on partial differentiation and complex numbers are restricted in scope. Here, as in the chapter on differential equations, the treatment falls short of the requirements of Part I of the degree course. The linear differential equation with constant coefficients is solved only when the right-hand side is zero. In the last chapter, on statistics, only the calculation of mean and standard deviation is dealt with; there is a brief mention of the normal distribution, a fuller treatment with an introduction to probability being promised in Vol. II.

In general the precision of statement and proof of fundamental results is reasonable for the H.N.C. course. An exception is the definition of Δs as the length of " *AB*, the (straight line) chord " with the statement that " the ratio of the chord $AB = \Delta s$ to the differential element of arc $AB = ds$ tends to unity ". The binomial series is given without proof on p. 14 but the validity of

the expansion is clearly defined. The only misprints noted are: § 29·4, line 7, d/dx should read $d/d(ax)$; § 34·3, Ex. 2, dy is omitted from the numerator.

C. G. P.

Mathematical Engineering Analysis. By RUFUS OLDENBURGER. Pp. xiv, 426. 45s. 1950. (Macmillan)

This book, although similar in title to a number of others which have been published in recent years, is most unusual in character. There are many works which purpose to introduce the engineer to various mathematical techniques which may assist him in organising or solving the equations with which he is confronted. In this work, however, the emphasis is placed on the initial formulation of the equations in problems which are mainly of engineering undergraduate standard. The author's aim is to assist the reader in "expressing physical situations in the form of mathematical relations".

This common difficulty may be due to a woolly physical appreciation of the problem which defies expression in the precise language of mathematics, or it may be due to inadequate command of the language itself. This book does not attempt to remedy any deficiency due to the latter cause and presupposes considerable fluency in advanced calculus.

The chapters are grouped under five major headings: Mechanics of Rigid Bodies, Electricity and Magnetism, Heat, Elasticity and Fluid Mechanics. Each is prefaced by an economical statement of fundamentals which, except in the case of the first section, is developed with considerable generality. Line, surface and volume integrals, vector notation and the notion of matrices are used when they serve the author's purpose. A heterogeneous collection of examples then follows in which the relations between the variables are developed with copious detail.

Dimensional Analysis receives a self-contained mention in each section, but is not made an integral part of the text; this weakness is occasionally aggravated by the use of equations which are dimensionally unsound. In a book of such wide scope it would be churlish to mention many significant omissions or to expect each section to be written with uniform authority.

The author, once a pure mathematician, eventually moved into the industrial field, and this book is no doubt based on his recollection of that journey. The trials and pitfalls he encountered may well trouble those who are about to tread the same path and to them it may prove a useful guide. Those who travel by other routes find other difficulties and to them this volume may be of value, not so much as a guide, but as a book for the journey.

D. A. J.

Geometrical and Mechanical Drawing. By H. H. WINSTANLEY. Pp. 251. 12s. 6d. (Edward Arnold & Co.)

This work is intended for the use of students preparing for the Matriculation Certificate Examinations of London University and the Northern Universities.

The teacher of this subject is faced with a range so wide that some aspects are often left out. In this work, no essential matter is omitted.

The author has shown considerable skill and patience in presenting the subject; this is especially true of the section on Solid Geometry—often of some difficulty to the young student.

The text and drawings are clear and well set out, although in a few instances the pages seem rather crowded, e.g. pages 75 and 134. The whole book is interspersed with questions from examination papers. Although no index is provided any item can quickly be found in its appropriate group in the Table of Contents.

The author is to be congratulated on providing an excellent course of study for the intending engineer and architect.

The book will prove a useful book of reference for more advanced students.

F. T. D.

Les Nombres et les Espaces. By G. Verriest. Pp. 188. 200 fr. 1951. (Colin, Paris)

This is a very readable little book which gives an account of some fundamental mathematical ideas, and does it in a charmingly elementary manner. There are five chapters and a brief final note. The first chapter deals with numbers and sets, including some of the paradoxes of the infinite, the second deals with Projective and non-Euclidean Geometry and some possible astronomical consequences. The third is on groups, the fourth on other elements of abstract algebra, and the fifth reverts again to geometries.

It would be useless to attempt to explain the content of the book in any detail. Those who already know what is there would hardly be interested, and those who do not would be best referred to the text; the reviewer does not believe that any alternative explanation would be either shorter or clearer than the author's. One striking feature of the book is the simple way in which the author has succeeded in conveying some of the ideas and results of the Galois Theory.

The level of the exposition is such that, although the contents of the book might not all be familiar to the intelligent mathematical graduate, the book could easily and profitably be read, and understood, by a reasonably intelligent sixth-former. It will certainly be of interest to many who, while they have no advanced mathematical training, are interested in learning about mathematical ideas.

D. B. S.

Mathematics, queen and servant of science. By E. T. Bell. Pp. xx, 437. 21s. 1952. (Bell)

Of those who write about mathematics for the intelligent non-mathematician, few are as lucid and precise as E. T. Bell, and none more stimulating. The present volume revises and amalgamates two earlier books, now out of print, *The queen of the sciences* and *The handmaiden of the sciences*. It is not a history of the subject, though there is much about the history of mathematics; rather it attempts to make plain to the layman the spirit of modern mathematics, its roots in the past, its sudden turns and spurts on the urgings of a Newton or a Gauss, its remoteness from the prevalent crude materialism and its extraordinary knack of becoming vitally relevant to material concerns. In many respects the book would replace the author's big *Development of mathematics* for those for whom the necessary technicalities of the latter work are too forbidding. The present book is more liberally illustrated from physics and astronomy, and the author remarks that "the material is about equally divided between pure and applied mathematics. The two are inseparable". Nevertheless, I would guess that the author's heart is with the mathematics, not with its applications.

The author is justifiably pleased that the two earlier books were read by many non-mathematicians—lawyers, doctors, engineers, business men, writers—anxious to find the spirit of mathematics, and he is even more pleased to have given numerous young readers a glimpse of what lies beyond the school curriculum. The new book should be equally successful; there is just as much need for its stimulating vigour. When Bell refers to the days "before mathematical instruction was denatured to meet the increasing demands of mediocrity", when W. D. Reeve talks of the educational system being "geared to the production of mediocrity", we in this country are of

course in the main not well enough informed about the educational problems of the U.S.A. to make useful comment, but we should be foolish to ignore the implied warning. To recognise the bright young mathematician, to train and encourage him, to urge him forward so that he may take his part in the advancement of knowledge, is still a large part of the duty of the teacher. The teacher who has read Bell's book, who has seen that it is on the library shelves, who has put it into the hands of his promising pupils, is taking his duty seriously.

T. A. A. B.

Conformal representation. By C. Carathéodory. 2nd edition. Pp. ix, 115. 12s. 6d. 1952. Cambridge Tracts, 28. (Cambridge University Press)

This, but for one new chapter, is a corrected version of the 1932 edition, the central purpose of the tract being to give a concise but comprehensive account of Riemann's theorem, that a simply-connected domain can be conformally mapped on a circle. The new chapter deals with the uniformisation theorem of Poincaré and Koebe, that a Riemann surface can be conformally mapped on one of (*a*) a sphere, (*b*) a euclidean plane, (*c*) a circle. Carathéodory's account pivots about a theorem on the topology of a Riemann surface, due to van der Waerden, This permits of great brevity, but the argument is austere and demands very close attention.

The corrections and the new chapter were completed shortly before the death of the distinguished analyst in 1950. The tasks of revising the English and of seeing the tract through the press were carried out by Mr. G. E. H. Reuter. As a self-contained account of the theory of conformal mapping, the tract remains essential to the serious student.

T. A. A. B.

Senior Mathematics. By J. J. de Kock and A. J. van Zyl. Pp. xv, 371. 12s. 6d. 1951. (Maskew Miller Ltd., Cape Town)

This book is a sequel to the "New Junior Mathematics" by the same authors, and completes the secondary course for schools in South Africa. It is a book written personally to the student, explaining the objects in the different sections, interesting him with the historical development, and being outspoken about which parts are mathematically essential, and which parts are needed to conform to an examination syllabus.

It is like the previous book in dealing with Arithmetic, Algebra, Trigonometry, Geometry, in that order, but there is much interplay, and stress is put on the place each has in the subject mathematics. No suggestion is given as to the order in which the sections in the different parts should be read.

The first section is on logarithms and their applications. The chapter on logarithms itself is long, and covers too much ground for it to be done at one reading, including as it does their use in calculation and their use in solving equations with indices. Applications are then given to mensuration and to interest and annuities. This latter chapter quotes and uses complicated formulae on compound interest and annuities, and seems to involve ideas which are mature for students at school, though explained in an interesting way.

In algebra emphasis is on the quadratic function, and starts with an interesting section on functional relationship. Here, as in other places in the book, much use is made of questions in which one word is to be inserted. The functions are illustrated by their graphs, and their variation and maximum and minimum values are considered before any attempt is made to solve equations. For such solution the order chosen is by graphs, by completing the square and formula, and lastly by factorising. It is explained that although the last method may be easy it is of little practical value, since few quadratic functions will factorise with rational coefficients. More complicated

factors and fractions are put late in the section, with the statement that their study is unnecessary except as pure manipulation for examination purposes. The chapter on simultaneous quadratic equations again starts graphically, and opportunity is here taken to introduce analytically the conic sections. The algebra is completed with a chapter on calculus, where it is a pity that dy/dx is used to represent the gradient of both the chord and the tangent to a curve.

The trigonometry is concerned with the extension of definitions to obtuse angles and to the sine, cosine, and area rules. At the beginning the extension of the definitions is not very clear, and with so much graphical work and use of coordinates in the previous part, the definitions might have been given as x/r y/r, y/x.

For geometry there is a good historical introduction, and the authors make a real effort to teach deductive methods. There is a chapter on clear thinking, with wordy questions which demand deductions, and which are taken from all spheres of life. The geometry itself starts with a list of axioms which are to form the basis, and these include many formerly known as theorems. There follow theorems, limited in number but more than those in the syllabus now in use in this country. Nearly all the facts which are here proved have been met informally in the previous volume. In examples there are both practical and theoretical illustrations of the deductive method. There is thus much of the systematising stage of geometry, combined with illustrations which keep alive the interest of the student, and help him to realise the thought behind the process.

The book as a whole would be found rather hard in our secondary schools, being above the level of our elementary mathematics, and below " additional " mathematics of the " O " level in our Certificate examinations. It does not contain as many examples as some teachers would like, but it does combine in an unusual way relation to life as it is and has been, and the development of mathematical methods, and many students would gain in inspiration more than they would lose by a lack of drill examples. Teachers would certainly gain ideas from it, and it is fair to end by saying that this book, and the junior book are invigorating and put into practice many of the ideas which Mr. van Zyl brought forward in his book "Mathematics at the Cross-Roads ". K. S. S.

Plane Analytic Geometry. An Intermediate Course. By A. B. SHAH and M. APTE. Pp. vi, 252. Price Rs. 5–8. 1951. (A. B. Shah, Poona)

This book presents a unified treatment of Conic Sections developing the ellipse, hyperbola and parabola from their common focus-directrix property. Rectangular Cartesian co-ordinates are used throughout and only modest demands are made on the student's knowledge of Differential Calculus.

A pleasing feature lies in the numerous worked examples and the plentiful supply of exercises—over 300—to which hints and answers are given. The treatment is in general compact, side issues being avoided but it is somewhat uneven in clarity.

The weakness of this work is to be found in the large number of errors. Many of these are concerned with trivial details of typography, the standard of the printing and diagrams being much below that to which we are accustomed. Unhappily some errors are more serious. The illustrative example on pp. 79–80 which achieves five errors in two lines is the worst the reviewer discovered but the unfinished example on p. 28 where a term AB is used with two entirely different meanings (product of two coefficients and length of a line) in the same set of equations is more likely to mislead the unwary student. The rather confusing note on identification of angle bisectors (pp. 34–35) is illustrated by an example in which one bisector conveniently passes through the origin.

A carefully revised and corrected second edition could be recommended as a useful inexpensive supplementary text-book and source of exercises. J.K.

Analytical geometry. By S. Suryanarayana Iyer. Pp. viii, 484. Rs. 12·12. 1951. (St. Joseph's College, Trichinopoli)

This is a successful attempt to collect into one text-book all the applications of the analytical method to straight lines and conics, and is suitable for students aiming at Scholarship level or higher. It treats of straight lines and line-pairs ; of the general conic before the circle, parabola, etc.; of tangential equations ; of general homogeneous co-ordinates ; and of invariants, covariants and contravariants. The bookwork is presented with a nice balance between completeness and economy, and most of the familiar topics are exhaustively discussed. The worked examples are carefully chosen to exhibit the effectiveness and beauty of the analytic method. The author usually indicates any better solution available by other methods, but chooses an unfortunate example to illustrate the equation of the normal in polars, where a page of analysis can be replaced by two sentences of pure geometry. There is a good selection of examples, many of them from the question papers of Universities whose identification is left as an exercise for the reader.

The printers, who do not claim to specialise in mathematical text, are to be complimented on their effort, although there remain a number of unimportant misprints which will doubtless be corrected in later editions. The use of Σ to represent summation as well as the tangential conic equation in the same phrase is disconcerting, and the comma could with advantage be omitted from the clumsy symbol $\Sigma_{1,2}$. The use of $x/a \sec \theta$ for $x \sec \theta/a$ as on p. 252 is also unfortunate.

The book is a welcome addition, if only as a work of reference, to the literature of this subject, and its qualities should recommend it to the student for whom it is intended, and to his teachers. W. J. H.

Intermediate geometry. By L. J. Lacey. Pp. xii, 363. 10s. 1951. (Macmillan)

This book is intended to break down for Advanced and Scholarship level pupils the barrier between Euclidean and Analytical Geometry, but it still appears to be a text-book on Rectangular Cartesian Geometry applied to straight lines and conics (in standard form) with three chapters of Euclidean Geometry added, and as such is adequate to its purpose. The last three chapters, for specialists, treat the general conic, line co-ordinates and envelopes, and related figures. Examples are well-graded and cater for pupils of varying abilities. The book-work is generally well presented, but suffers from more or less serious blemishes. Pupils at this stage deserve a better definition of locus than " the curve traced out by a variable point which is constrained to move . . ." and of the equation of a locus than " the equation which is satisfied by every point on the curve and by no other point ". The treatment of the " perpendicular form " of the straight line is open to objection, and arises from the idea that the canonical form can be $-x \cos \theta - y \sin \theta = p$ as well as $x \cos \theta + y \sin \theta = p$, an error induced by the faulty conclusion that if $\tan \theta = m/l$ then $\cos \theta = l/\surd(l^2 + m^2)$. The symbol $AB.CD$ to denote the meet of AB and CD is a handy contraction, but it is out of place, without explanation, in a paragraph such as 5·35 where it has also the usual meaning of the rectangle contained by the two lines.

The book would be improved by the removal of such errors as (the italics are mine) : (p. 108) the gradient of the tangent at P is given by the value of *the differential coefficient* at that point ; (p. 111) the interchanging of lines 5 and 6 from the bottom of the page ; (p. 122) the naive assumption that " the

general equation of a circle " has unity for the coefficients of x^2 and y^2; (p. 257) the intersection of $\frac{x}{a}\sec\theta \; \frac{y}{b}\tan\theta=1$ (*sic*) and $\frac{x}{a}\tan\theta-\frac{y}{b}\sec\theta=1$ is (a, b); (p. 303) the mathematically shocking statement that " the equation of the lines enveloping the parabola $y^2=4ax$ is $am^2=nl$ " and two similar mis-statements for the ellipse and the hyperbola. W. J. H.

Advanced Five-figure Mathematical Tables. By C. Attwood. Pp. v, 69. 4s. 6d. 1951. (Macmillan)

This handy volume gives, for the first time in a readily accessible form, 5-figure tables of circular, exponential and hyperbolic functions at close intervals. It also contains ratios of angles in degrees and decimals, the factorial function $\Gamma(x+1)$, elliptic integrals, and notes on interpolation. The tables are pleasant to look at, and easy to read, but some type is out of line on p. 41. When the integer changes, the number is given in full. The author appreciates the limitations of mean differences, and uses various devices to give correct values. The Normal Curve is given, but when will someone publish a cheap set of statistical tables? G. A. G.

A new geometry and trigonometry. By A. R. Bielby. Pp. viii, 448. 10s. 6d.; without answers, 10s. In two parts, 6s. 6d. each. 1951. (Longmans Green)

This book covers all Ordinary Level requirements in the General Certificate of Education in pure geometry and numerical trigonometry. Proofs of all theorems are given, and numerous test papers are interspersed between groups of chapters.

Each part constitutes a two-year course; in the first part the geometry covered includes angles, congruence, quadrilaterals, similarity, the theorem of Pythagoras and intercepts. The sine, cosine and tangent of an acute angle are introduced, and problems on right-angled triangles are done, using the 3-figure tables provided. In the second part the usual geometry course is completed, with revision of Pythagoras' theorem, and a chapter on problems in geometry, which groups the results of part I in a new way. 4-figure tables are used, and the sine and cosine rules are developed. A large section is devoted to three dimensions, and plan and elevation methods are introduced.

The book contains many excellent features and is clearly the careful work of an experienced teacher. For instance, the author is at pains to explain the technical terms of the subject most carefully, and in the excellent " chapter summaries " lists of new words are included. The book abounds in practical illustrations of the various topics, and the examples are most varied and well graded. An excellent chapter on constructions, mostly on copying triangles, precedes the chapter on congruence, in which the abbreviations S.A.S., A.A.S., R.H.S. are used. The book encourages sensible abbreviations, but I think the author is mistaken in using them in the revision papers. Much care is taken over the introduction of similarity, which is done earlier than is usual. before the study of quadrilaterals or the theorem of Pythagoras. The idea of definitions is carefully explained, as are those of tangency and loci. It is good to see that students are encouraged to try to measure to the nearest $\frac{1}{100}$ inch in their geometrical drawing.

Attention needs to be drawn to one or two minor defects in the book. To suit schools where trigonometry is begun later, a fuller summary of the trigonometrical functions should appear at the beginning of Part II. No list or index of theorems proved in the book is given, and the proofs themselves could be given more prominence. The idea of the locus appears rather late, as does the study of areas, neither of these topics appearing in Part I. There

is an attempt to distinguish between " gradient " and " incline " which may well prove more misleading than helpful, because the author's ideas are not generally accepted in practice.

On balance the book can be thoroughly recommended and deserves wide circulation. The printing, both of text and of diagrams, is excellent.

F. J. T.

Leçons de logique algébrique. By H. B. Curry. Pp. 163. 1952. Collection de logique mathématique, series A. (Gauthier-Villars, Paris)

An algebra of logic is defined by the author as a free variable formal system, containing no bound variables, organised into five categories, comprising the " alphabet " of the system, the elementary propositions, the axioms, the rules of deduction and the theorems. The algebras considered in this book, in order of increasing complexity, are classical lattices, implicative lattices without negation, and implicative lattices with one of four types of negation. There are also brief references to modal algebra and combinatorial algebra.

The most original part of the book is the first chapter with its analysis and description of the nature of a formal system. The evolution of the concept of a formal system is traced through three stages of development. In the first stage the system is regarded as an aggregate of propositions *about* certain intuitively given notions, some propositions, the axioms, being held to be self-evident and others being derived from the axioms by some more or less unspecified laws of reasoning. In the second stage of development the " laws of reasoning " take their place alongside the axioms as an explicitly formulated set of derivation rules. In the final stage the derivation rules, like the axioms themselves, are recognised as arbitrary conventions, not formalisations of natural laws, the elements are freed entirely from their intuitive significance and the proposition itself becomes a formula, that is, a string of elements satisfying only certain rules of formation.

Professor Curry's conception of a formal system is in some respects different from the original conception of Hilbert's and he is at some pains to make this distinction clear. For Hilbert a formal system was an assemblage of signs, L, subject to certain transformation rules which were formulated in a secondary system, M, the elements of which were the formulae of L. If M itself were completely formalised then the descriptions of operations in M required a tertiary system N, and so on, so that in practice the meta-system M was only partially formalised. Curry regards a formal system as a part of an ordinary language of communication U (like the English language); this formal part introduces new symbols into the language U and itself forms a new language (which Curry denotes by A). Language A consists of a set of names for the otherwise unspecified elements of the system and signs for the operations and predicates of the system. Language A cannot itself express the transformation rules which are formulated in U (with or without the help of a new technical vocabulary). The significance of the names of the language A, that is, their relation to be body of nouns of the language U, is not required to be specified. The object of this is, presumably, to escape the charges which have been raised against Hilbert's formalism that it reduces mathematics to marks on a piece of paper; in Curry's formalisation of arithmetic the numbers are, one supposes, *any interpretation of the names of the system consistent with the axioms.* This is of course in keeping with the traditional practice in geometry in which *point*, *line* and *plane* are said to be any objects which satisfy the axioms of the geometry, but it has the grave disadvantage of leaving the mathematician in a vicious circle of concepts. For if none of the entities of mathematics is definite, then there are no *objects* which in fact satisfy the axioms. Curry anticipates this objection by suggesting that the names which are the elements

of a formal system may simply be taken to be the names of (other) signs. Thus for instance in a formal system in which 0, 0′, 0″, ... are names, we may take them as names for the signs " * ", " * ! ", " * !! " and so on. This seems to the reviewer to indicate a misunderstanding of the character of a (written) language ; although some of the nouns in a language are names of particular objects, like Winston Churchill or Isaac Newton, other nouns, like " negation ", are not ; if now I introduce some sign like " $\sim$ " and say that negation is the name of this sign then in fact all that I am doing is to give two signs *equivalent* roles in the language. In brief, numbers are not the marks 0, 0′, 0″, and so on, nor are they any objects which these marks may be held to name, but they are the *role* which the signs 0, 0′, 0″, ... play in a system. The signs themselves, 0, 0′, 0″, or *, * !, * !!, ... are just the actors which have a certain part in the play, not the play itself.

Towards the end of the section on formal systems Curry introduces the important notion of the epi-theorems of a formal system, which are general theorems *about* the propositions and proof processes of the system itself. Assuming that the epi-theorems have not been formulated in a completely formal system their method of demonstration may well transcend the proof processes of the system and the only limitation on these methods is that they must provide a definite procedure for determining in the system the proofs of the particular propositions of the formal system which are instances of the epi-theorems. Curry remarks that the epi-theorems are the life and soul of formal methods. This remark throws a new light on the definition given by the author in his book *Outlines of a formalist philosophy of mathematics* (written in 1939 but published this year), a definition repeated in the work under review, that mathematics is the science of formal systems, and understood in the form " mathematics is the science of the *epi-theorems* of formal systems " the definition is far more acceptable.

In preference to the traditional formulation of the theory of implication by means of propositional variables p, q, r, etc., and a set of axioms for implication such as

$$p \to (q \to p),\ \{p \to (q \to r)\} \to \{q \to (p \to r)\},$$

and

$$\{p \to (q \to r)\} \to (\,(p \to q) \to (p \to r)\,),$$

Curry introduces implication as an operation in a lattice of names ; this operation of implication is denoted by " $\supset$ " and is regarded as an operation which constructs a new *name* by joining two names (and so designates an operation which joins two elements to form a new element). In addition two further implication symbols are employed, the sign " $\leqslant$ " for an implication predicate (for forming sentences out of names) and the sign " $\to$ " as an implication connective between sentences. This last implication sign is regarded as having the familiar intuitive significance of " if ... then ", but both the implication operator and the implication predicate are treated purely formally. Using Russell's assertion sign " $\vdash$ " as a predicate of names, Curry shows that by substituting " $\vdash x \supset y$ " for " $x \leqslant y$ " his relational algebra of propositions may be brought into the traditional logistic form, and conversely, if the lattice contains a special element 1 satisfying the predicate $x \leqslant 1$ for all elements x of the lattice a propositional algebra (without negation) may be given the form of a relational lattice by substituting " $a \leqslant b$ " for the assertion " $\vdash a \supset b$ " and " $1 \leqslant a$ " for " $\vdash a$ ".

A lattice with the predicate " $\leqslant$ " and the operators " $\supset$ " and " $\wedge$ " is said to be an implication lattice if the operators satisfy the postulates

$$a \wedge x \leqslant b \rightleftarrows x \leqslant a \supset b.$$

It is shown that every implication lattice is distributive and that every

distributive lattice is implicative if it is finite. The dual of the implication operation in an implication lattice is defined to be the difference operator satisfying the postulates

$$a \leqslant b \vee (a - b),\ a \leqslant b \vee x \rightarrow a - b \leqslant x,$$

where " $\vee$ " is the dual of " $\wedge$ ". A lattice with the predicate " $\leqslant$ " and the operators " $-$ " and " $\vee$ " is called a subtractive lattice and it is proved that a subtractive lattice is distributive and that a finite distributive lattice is an implication lattice.

Curry describes four different ways in which negation may be introduced into an algebra. In the first of these we use some arbitrary element f in an implication lattice and define a negation of a, $\neg a$, to be equivalent to $a \supset f$. This species of negation is called *refutability*. The second species of negation, called *absurdity*, is defined in a lattice which contains a special element 0 satisfying $0 \leqslant x$ for every element x, by taking $\neg a$ to be equivalent to $a \supset 0$. Neither of these species of negation necessarily satisfies the law of excluded middle. If we postulate the law of excluded middle, we obtain two further species of negation, the so-called strict negation, which is refutability plus the law of excluded middle, and classical negation, which is the absurdity theory of negation in a system postulating the excluded middle. The second species of negation gives rise to intuitionistic algebra, that is, Heyting's formulation of intuitionistic logic. It is interesting to note that Gödel has proved that intuitionistic logic is no more intrinsically finitist than classical logic. A non-classical logic is not the same thing as a three (or more) valued logic, and in fact we are no better able to represent a formalised theory of implication in a many-valued logic than in the familiar logic of two values. Curry wrongly attributes (p. 138) the first clear conception of truth tables to Post (1921); in fact the first formulation in modern times—for the notion was familiar to the Stoics—is to be found in Wittgenstein's *Logisch Philosophische Abhandlung* published during the first World War.

After a final chapter on *secondary algebras*, which are formalised meta-algebras whose elements are the epi-propositions of the primary algebra, the book concludes with an appendix on Lukasiewicz's bracket-free notation for certain types of formal systems and the representation of systems in terms of two symbols only (by analogy with the construction of numerals from the two symbols " 0 " and " ′ "). R. L. GOODSTEIN.

Outlines of a formalist philosophy of mathematics. By H. B. CURRY. Pp. vii, 75. Fl. 7.50. 1951. (North-Holland Publishing Co., Amsterdam)

Professor Curry's contribution to this new series of books on logic and the foundations of mathematics is as sharp and clear as an outline should be but it is a statement of a personal point of view which, despite a plea for more tolerance in the foundations of mathematics, finds little virtue in any outlook but its own.

The author explains in the preface that the manuscript of the book was prepared in 1939 and represents the views which he then held, not the views which he would defend today. Speculation about what these present views may be is profitless but there seems little doubt that Professor Curry's present standpoint must be substantially that of the monograph or he would otherwise surely have hesitated to permit its publication.

The thesis of the book is that mathematics is the science of formal systems; it is a thesis that is postulated rather than defended, and it is probably intended to be taken more as a working hypothesis than a formal definition of mathematics. It seems to the reviewer that it would be a little nearer to the truth to call logic, not mathematics, the science of formal systems. A formal

system is only the skeleton of mathematics, something that helps to make the body stand upright, not its heart, or lungs, or its *élan vital*. The science of formal systems is no more the whole of mathematics than the science of letters is literature or the science of metre, poetry. Composition, invention, definition, the discovery of hidden connections and unperceived differences, above all *symbolism*, are aspects of mathematics that lie outside formalism and give to mathematics its character, vigour and importance.

Professor Curry holds that formal systems are to be judged by their acceptability, that is, by the extent to which they fulfil their ordained purpose. Thus he says that so long as its usefulness persists, classical analysis needs no other justification whatever, and by the same token intuitionistic analysis is utterly useless because it is so complicated. The weakness of this argument is that it ignores the possibility—a possibility which has become fact in the years since the monograph was written—that there may be a finitist system of analysis no less acceptable to modern physics than the classical theory of functions. The choice between classical and finitist analysis, if a choice is to be made, must therefore rest on grounds other than acceptability, and I would suggest that the discoveries of the past twenty years show that these grounds are to be found in the *interpretation* of formal systems.

In the chapter on syntax it is suggested that the time has come to sign a declaration of independence from the domination of the syntactical standpoint in the analysis of language. It is easy to sympathise with the proposal but the reason for which it is advanced scarcely seems to be adequate. It is argued that the thesis that concept words are dispensable cannot be established by the now familiar process of translating into the syntactical mode of speech because, sometimes, the translation does not have the same meaning as the original. As an example of this failure, the sentence

(S) *Seven is a number*

is compared with the transcription

(S^*) " *Seven* " *is a number word.*

By translating (S) and (S^*) into German it is shown that these sentences do not have the same meaning. For the translation of (S^*) which is taken to be

(S'^*) *in der Englischen Sprache ist* " *seven* " *ein Zahlwort*

refers to a fact of the English language to which (S) itself makes no reference. Certainly the translation of (S^*) is not

" *Sieben* " *ist ein Zahlwort*

since the sentence (S^*) is about the sign " seven ", not the concept seven, so that " seven " is invariant under translation, but the attempt to translate (S^*) by

" *seven* " *is a Zahlwort*

is rejected as meaningless in a German context. I am quite unable to accept the view that " ' seven ' is a Zahlwort " is meaningless in a given context, but I think that we find a complete rejoinder in the observation that, if we are taking more than one language into account, the transcription of (S) in the syntactical mode of speech is not (S^*) but rather

" *seven* " *is an English number word*

of which the translation is

" *seven* " *ist ein Englische Zahlwort*

which clearly has the same meaning.

Whether the empirical formalism which it defends is acceptable or not, this volume is important for its analysis of the idea of a formal system and the steps which it takes to set mathematical formalism free from all philosophical

preconceptions of the nature of mathematics. Both the author and the North-Holland Publishing Company are to be congratulated on producing a little book which delights the eye as much as it stimulates the mind.

R. L. GOODSTEIN.

The propositional logic of Boethius. By K. DÜRR. Pp. x, 79. Fl. 8. 1951. Studies in logic and the foundations of mathematics. (North-Holland Publishing Co., Amsterdam)

This work is a translation by N. M. Martin of a hitherto unpublished manuscript written in 1939. It studies the rules of inference recorded or discovered by Boethius, the last of the ancient logicians, who wrote on the hypothetical syllogism (propositional logic) between 510 and 523 A.D., and whose work exerted a great influence on the logic of the middle ages. Professor Dürr discusses first the sources of Boethius' work and rejects the view that Boethius was directly influenced by the great Stoic logician Chrysippus. This is followed by a detailed analysis (using the notation of the Polish school of logic) of the eight classes of rules of inference which Boethius describes. The analysis is somewhat complicated by the uncertainty which exists about the precise significance of implication in these rules, and Dürr tests each rule on the twin assumptions of material and strict implication, finding the majority of them "correct" on either assumption. The book concludes with an appendix by the translator which supplements some of the formal details of the text.

R. L. GOODSTEIN.

Enzyklopädie der mathematischen Wissenschaften. Vol. I 1, Part I, 11. *Mathematische Grundlagenforschung.* By A. SCHMIDT. 1.34. Pp. 48. 1950. (Teubner, Leipzig)

This article on the foundations of mathematics was written in 1939, but contains two short sections and a number of footnotes added during printing in 1949. It presents a summary of methods and results in general foundation problems and the foundations of number theory, and is not concerned with the foundations of any special branch of mathematics like Analysis or Topology.

The article starts with an account of the nature of formal axiomatic systems and the problems of their completeness and freedom from contradiction. This is followed by a description of recursive number theory, and an account of Gödel's method of Arithmetisation of Syntax, leading up to proofs of the two Gödel theorems that a formal system which admits an arithmetisation of its syntax necessarily contains undecidable questions, if it is free from contradiction, and this freedom from contradiction cannot be established by the resources of the system itself. In a later section the fundamental part which multiplication plays in this result on the impossibility of an intrinsic proof of freedom from contradiction is explained, and it is shown that formal systems which stop short at addition are both complete (that is, contain no undecidable questions) and demonstrably free from contradiction. This section concludes with an account of Gentzen's extrinsic proof of freedom from contradiction of number theory based on a transfinite induction for ordinals less than ϵ (the limit of ω_n, where $\omega_0 = \omega$, $\omega_{n+1} = \omega^{\omega_n}$). The third section deals with the Frege-Russell theory of natural numbers, the theory of types and the problems which arise in the attempted reduction of mathematics to logic. The last section is devoted to Brouwer's intuitionist mathematics with a footnote reference to other finitist systems of analysis.

R. L. GOODSTEIN.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

and the
oducing a
d.
ODSTEIN.

8. 1951.
-Holland

ed manu-
iscovered
pothetical
ose work
ssor Dürr
Boethius
s followed
gic) of the
nalysis is
e precise
the twin
of them
pendix by
text.
ODSTEIN.

art I, 11.
8. 1950.

1939, but
printing
oundation
rned with
Topology.
systems
on. This
ccount of
f the two
ion of its
n contra-
d by the
art which
proof of
l system
decidable
conclude
diction of
an ϵ (the
he Frege
ms which
section i
ference t
ODSTEIN

MATHEMATIC

Treasurer's Statement for th

1951	RECEIPTS		1952
£115 1 11	Balance, November 1st, 1951		£577 19
	Subscriptions for 1952 :		
2,121 18 0	Ordinary Members	£2,140 1 2	
20 3 0	Junior Members	19 8 6	
			2,159 9
	Subscriptions for previous years :		
261 17 6	Ordinary Members	89 13 0	
	Junior Members	0 15 0	
			90 8
	Subscriptions in advance :		
42 7 0	Ordinary Members	72 9 0	
9 19 6	Junior Members	9 12 0	
			82 1
135 1 0	Life compositions		73 10
395 10 11	Proceeds of sales of Gazettes		425 7
417 13 9	Proceeds of sales of Reports		518 15
24 5 0	Advertisements in Gazettes		53 0
40 12 8	Interest on investments		43 19
41 5 0	Subletting of office		56 5
21 3 2	Sales of Isaac Newton documents		11 16
	Return of income tax on subscriptions		350 11
0 1 7	Sundries		11 9
£3,647 0 0			£4,454 12

1951	LIABILITIES	1952
£52 6 6	Subscriptions in advance	£82 1
1996 3 0	General fund, balance at this date	2084 8
£2048 9 6		£2166 9

Liabilities also include the amounts received from 231 members who have pai life subscriptions.

November 21, 1952 J. B. MORGAN, *Hon. Treasur*

MATICAL ASSOCIATION

nt for the Year ending 31st October, 1952

1952
.. £577 19 9
2
6
— 2,159 9 8
0
0
— 90 8 0
0
0
— 82 1 0
.. 73 10 0
.. 425 7 5
.. 518 15 5
.. 53 0 0
.. 43 19 6
.. 56 5 0
.. 11 16 5
. 350 11 0
.. 11 9 5
£4,454 12 7

1951	PAYMENTS		1952
	Gazettes Nos. 313–317 (September 1951—September 1952) :		
£1,462 0 1	Printing	£2,136 16 4	
224 13 7	Postage and despatch ...	196 17 2	
200 0 0	Editor	200 0 0	
37 17 9	Index	53 0 3	
			£2,586 13 9
	Reports :		
125 15 0	Reprinting First Geometry		
109 4 9	Algebra		
	Second Geometry ...	171 9 0	
	Mechanics	68 9 6	
8 14 6	Printing Calculus (completion)		
23 11 9	Rules of the Association ...		
			239 18 6
351 0 11	Expenses of Committees		276 18 7
1 13 0	Expenses of Problems Bureau		1 12 0
41 4 2	Expenses of Annual Meeting		21 15 5
	Grants to Branches :		
21 12 0	London (incl. arrears) ...	27 15 0	
2 2 0	Yorkshire	2 11 0	
	Bristol (incl. arrears) ...	5 8 0	
2 8 0	Manchester and District ...	2 8 0	
4 8 6	Cardiff	3 9 0	
4 11 6	Midland	3 18 0	
3 3 0	North Eastern	3 18 0	
2 12 6	Liverpool	2 8 0	
2 14 0	Sheffield and District... ...	1 14 6	
3 0 0	Southampton and District ...	2 5 0	
1 4 0	South-West Wales	0 19 6	
4 13 0	Leicester and County ...	1 19 0	
2 2 0	North Staffs and District ...		
1 4 0	Sydney, N.S.W.	0 13 6	
0 18 0	Queensland	1 1 0	
1 10 0	Victoria	2 14 0	
	South Africa	6 5 6	
	...		69 7 0
185 0 0	Clerical assistance		193 0 0
58 16 6	Stationery and printing		77 2 0
101 1 4	Office rent and expenses		142 11 0
50 9 3	Postage		43 4 4
27 8 7	Festival week		
	Membership List		32 2 9
2 6 7	Sundries		11 13 4
	Balance November 1st, 1952 :		
	Cash in hand	1 3 5	
	Cash in bank	757 10 6	
577 19 9			758 13 11
£3,647 0 0			£4,454 12 7

1952
.. £82 1 0
. 2084 8 3
£2166 9 3

1951	ASSETS	1952
£577 19 9	Balance in bank and in cash	£758 13 11
1,100 0 0	War Loan Stock	1,100 0 0
	Due from Messrs. G. Bell & Sons, Ltd., for sale of	
370 9 9	Gazettes and Reports	307 15 4
£2048 9 6		£2166 9 3

who have paid

Subscriptions are due from 105 members.

Assets also include Books in the Library, Gazettes, Reports and office furniture, of which no valuation has been made.

Hon. Treasurer

—Audited and found correct.

December 11, 1952 W. J. HODGETTS.